TWO DIFFERENT PERSPECTIVES ON WHOLE NUMBER ARITHMETIC: THE THEORETICAL CORE THAT MAKES DIFFERENCE

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Outline

1. The educational significance of whole number arithmetic
   整数算术的教育意义
2. Two traditions of whole number arithmetic
   两种不同的整数算术
3. The theoretical core of school whole number arithmetic
   小学整数算术的理论核心
4. The two cornerstones of learning school whole number arithmetic
   算术学习的两块基石
5. The concern of premature "algebra"
   过早引进代数的担忧

1. The educational significance of whole number arithmetic

-- a reflection on Piaget's ambition
整数算术的教育意义-- 从皮亚杰的雄心说起

During several decades, Piaget had a big impact on the field of mathematics education.

In 2008, however, the Final Report by the U.S.A. National Mathematics Advisory Panel announced that his "highly influential theory" has "consistently been shown to be wrong."

It is a shame that such a prominent figure exited the field in such a quiet way, without the serious discussion and reflection he deserves.

I would like to start with a brief reflection on Piaget's ambition, which may inspire our careful consideration of the educational significance of whole number arithmetic.

1. The educational significance of whole number arithmetic

-- a reflection on Piaget's ambition
整数算术的教育意义-- 从皮亚杰的雄心说起

Prehistoric man

A child's early life (0—12)

The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress (*that our species*) made in the logical and rational organization of knowledge and the corresponding formative psychological processes (*of a child*). (Piaget, 1969,p.4)

发生认识论的一个基本假设是：人类群体发展按照逻辑推理而组织知识的过程，和儿童个体与之相应的心理形成的过程是相似的。

the progress (*that our species*) made in the logical and rational organization of knowledge
人类群体发展按照逻辑推理而组织知识的过程

the corresponding formative psychological processes (*of a child*)
儿童个体与之相应的心理形成的过程

With this hypothesis, the most fruitful and the most obvious field of study (*of epistemology*) would be the reconstituting of human history – the history of human thinking in prehistoric man.

有了这个假设，认识论中成果最为卓越的认识论研究领域将是：重建人类的历史——史前人类的思想史。

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the corresponding formative psychological processes (*of a child*)
儿童个体与之相应的心理形成的过程

... we are not very well informed in the psychology of primitive man, but there are children all around us. It is in studying children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, and physical knowledge.

..... 关于史前人类的心理活动我们知之甚少。然而我们身边有的是儿童。研究儿童，便是我们得以研究史前人类逻辑知识、数学知识和物理知识发展的最佳途径。

Therefore, "the most fruitful and the most obvious field of study (*of epistemology*) would be the reconstituting of human history – the history of human thinking in prehistoric man". **[Piaget's ambition]**

所以，认识论中成果最为卓越的认识论研究领域将是：重建人类的历史——史前人类的思想史。[皮亚杰的雄心]

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The fundamental hypothesis:
There is a parallelism between the progress (*that our species*) made in the logical and rational organization of knowledge and the corresponding formative psychological processes (*of a child*).

基本假设：人类群体发展按照逻辑推理而组织知识的过程，和儿童个体与之相应的心理形成的过程是相似的。

The educational significance of whole number arithmetic
整数算术的教育意义

Not under any influence

the progress (*that our species*) made in the logical and rational organization of knowledge
人类群体发展按照逻辑推理而组织知识的过程

没有外在影响

Under the influence of adults (some kind of education)

在成年人的（某种教育）影响之下

the corresponding formative psychological processes (*of a child*)
儿童个体与之相应的心理形成的过程

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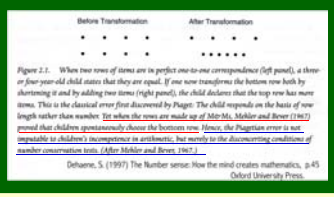
Other flaws of Piaget's theory

- Recapitulation Theory of Ernst Haeckel (Ontogeny recapitulates phylogeny), from which Piaget's fundamental hypothesis was derived, was discredited.
  皮亚杰个体思维发展复演人类思维发展的之假想的根据——海克尔的复演说被证明是不对的。
- Some of Piaget's classic research were disproved.
  一些皮亚杰的经典实验被证明是有问题的。



Dehaene, S. (1997) The Number sense: How the mind creates mathematics, p.45 Oxford University Press.

Romanes's 1892 copy of Ernst Haeckel's embryo drawings

However, Piaget's hypothesis is inspiring...
然而，皮亚杰的假想颇有启发性.....

人类数学史
The history of mathematics

Under the influence of adults

在成年人的影响之下

The process of individuals learning mathematics
个人学习数学的进程

Based on the discussion above, we may form the following hypotheses using common sense:
基于以上的讨论，我们是否可以从常识中归纳出以下两点：

1）To a certain degree, the process of a child developing mathematical knowledge parallels the history of the development of mathematical knowledge of our species. Thus, the history of mathematics may to some degree inspire our mathematical education. For example, the order of introduction of number types should mirror that in history. Whole number arithmetic, the earliest set of mathematical knowledge of human beings, becomes the start point of a child's math education.
当今儿童获得数学知识的进程，和人类数学知识发展的历史进程存在着某些可以类比的地方——例如在数集和学科引进的顺序上，数学史可能对于数学教育有着启示性。而人类最早的数学知识"整数算术"应该是当今人类个体学习数学的起点。

2）Because the process of a child attaining mathematical knowledge is guided by existing mathematical knowledge, the organization and presentation of mathematical knowledge will have a substantial impact on their learning.
由于儿童学习数学知识的进程是在已有数学知识的指引下进行的，那么已有知识的组织和呈现，可能对儿童的学习质量产生实质性的影响。

Based on the above two statements （基于以上两点，那么）：

1）Whole number arithmetic is significant for mathematical education.
整数算术在儿童的数学教育进程中有着非常重要的意义。

2）Given that there are multiple ways to organize and present whole number arithmetic, one that embodies the spirit of the discipline better would be more valuable in terms of students' learning mathematics.
如果存在若干种小学算术知识的组织和表现方式的话，越是接近数学精神的，于学生数学学习的价值越高。

2. The two traditions of whole number arithmetic
两种整数算术

1. Primitive whole number arithmetic 原始整数算术
   a) Arabic numeral system 阿拉伯数字
   b) The algorithms of the four operations 四则运算法则
2. School whole number arithmetic 小学整数算术
   a) Arabic numeral system 阿拉伯数字
   b) The algorithms of the four operations 四则运算法则
   c) A mathematical theory with the model of *The Elements*
      以《原本》为参照建立的小学算术理论

The efforts to construct the theory of school arithmetic

In the mid-nineteenth century, with the movement toward public education in the U.S. and Europe, mathematical scholars participated in producing elementary school arithmetic textbooks. Their exemplar was Euclid's *Elements*, the most influential mathematics textbook in history. These scholars followed the approach of *The Elements*, striving to establish a mathematical theory of school arithmetic. Near the end of the nineteenth century this theory was close to complete.

Ma, 2013, A Critique of the Structure of U.S. Elementary School Mathematics

A correspondence between *The Elements* and the theory of school arithmetic
小学算术理论与《原本》的元素之对照

The Elements 《原本》		Theory of school arithmetic
• Definitions 定义	←---→	• Definitions
• Postulates 公设	←---→	• Principles (e.g., number of same unit...)
• Common Notions 公理	←→	• Laws (e.g., commutative law...)
• Propositions 命题	←---→	• Statements (e.g., inverse operations...)

The efforts to construct the theory of school arithmetic

3. The theoretical core of school whole number arithmetic
小学整数算术的理论核心

- One basic concept – unit （一个基本概念：单位一）
- Two basic quantitative relations:（两种基本数量关系）
  – The sum of two numbers（两数之和）
  – The product of two numbers（两数之积）

The basic concept of Unit

Definition of *Unit*
One, or a single thing, is called a *unit* or *unit one*.
A group of things, if considered as a single thing or one, is also called a *unit*, a *unit one*, or a *one*.

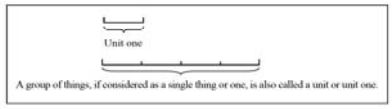


Fig. 1: The definition of unit

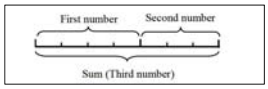
(Discussion: "Unit," indeed, is a self-evident concept for us. The definition of "unit" has three levels of abstraction: one thing; one; and a group of things considered as a single thing or one. These three levels of abstraction are the cornerstones from which the two basic quantitative relationships are derived.)

Definition of *Number*
A number is a unit (one) or a collection of units (ones).

The sum of two numbers

Definition of *the sum of two numbers*

The *sum* of two numbers is a third number which contains as many units as the other two numbers taken together.



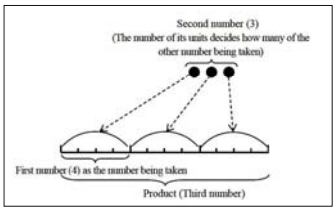
(Discussion: The definition of the sum of two numbers is derived from the definition of "unit" and that of "number." It, nevertheless, is only related to the first sentence in the definition of "unit: "One, or a single thing, is called a *unit* or *unit one*.")

Defining addition and subtraction

The product of two numbers

Definition of *the product of two numbers*

The *product* of two numbers is a third number which contains as many units as one number being taken as many times as the units in the other.



(Discussion: The definition of the product of two numbers is also derived from the definitions of "unit" and of "number." However, the second sentence in the definition of unit, "A group of things, if considered as a single thing or one, is also called a *unit*, a *unit one*, or a *one*," plays a critical role. The third level of abstraction in the definition of "unit" is reached now.)

Defining multiplication and division

Quantitative relationships in school arithmetic: from basic to sophisticated

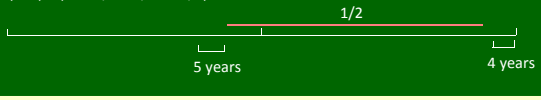
When students, learning school whole number arithmetic, tend to consider 3 + 2 as a sum, 5 – 2 as a difference, 3 × 4 as a product and 15 ÷ 5 as a quotient, they have attained the ability to analyze quantitative relationships, not only simple ones, but also relatively sophisticated ones. For example, (2 + 3) + (6 – 5), the sum of a sum and a difference, or, (20 – 2) × (3 + 1), the product of a difference and a sum. This ability, obviously, will prepare them well for moving on to higher-level subjects such as algebra.

An example of the sophisticated problems that school arithmetic can solve

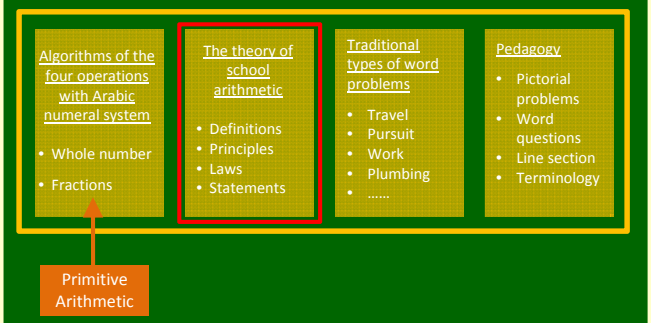
Diophantus' Epitaph (About 250 B.C. Greek mathematician)

This tomb hold Diophantus Ah, what a marvel!
And the tomb tells scientifically the measure of his life.
God vouchsafed that he should be a boy for the sixth part of his life;
when a twelfth was added, his cheeks acquired a beard;
He kindled for him the light of marriage after a seventh,
and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child,
when he had reached the measure of half his father's life,
the chill grave took him.
After consoling his grief by this science of numbers for four years,
he reached the end of his life.

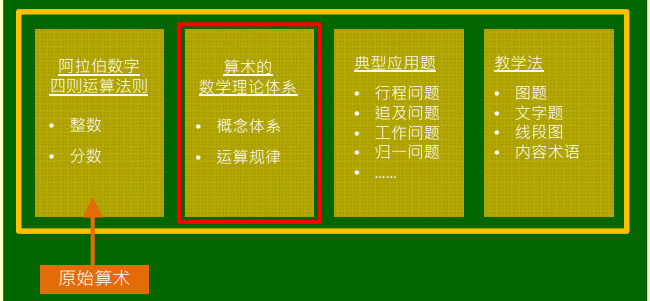
(5+4) ÷ (½ – 1/6 – 1/12 – 1/7) =



The four components of school arithmetic



小学算术学科的四个组成部分



4. The two cornerstones of learning school whole number arithmetic

学习整数算术的两块基石

1. Basic skills (mainly, but not only, from primitive arithmetic)
   基本技能
2. Basic concepts (mainly from the theory of school arithmetic)
   基本概念

Chinese culture has a tradition of paying attention to "foundations."

I consider that a main reason why China embraced, applied, enriched, and improved school arithmetic for more than five decades.

In the U.S., the efforts to construct the theory of school arithmetic fell apart during progressive education. The subject was abandoned in early sixties of last century.

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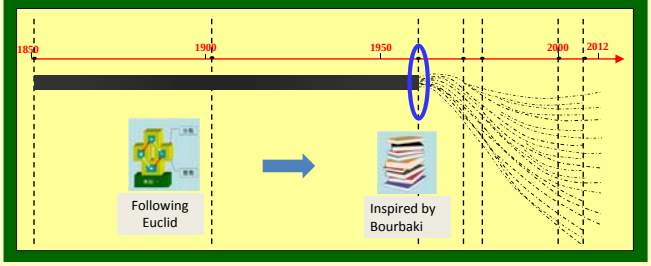

5. The concern of premature "algebra"

过早引进代数的担忧

- The trend of "pushing" algebra down to early grades of elementary school
  - Starting from the US
  - Motivation: middle school and high school students in the US are not able to do algebra
- Concern
  - Solutions:
    - A) To start teaching premature "algebra" early
    - B) To learn school arithmetic well and lay down a solid foundation for real algebra