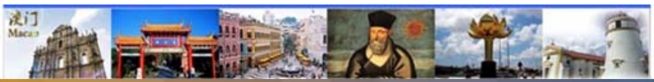
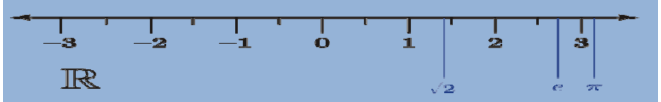


ICMI Study 23 - Primary Mathematics Study on Whole Numbers
Macau, China - June 3 - 7, 2015



QUANTITIES, NUMBERS, NUMBER NAMES, AND THE REAL NUMBER LINE

Hyman Bass

SCHOOL OF EDUCATION
UNIVERSITY OF MICHIGAN

PART I
QUANTITIES, NUMBERS, AND THE NUMBER LINE

PART II
NUMBER NAMES
BASE-10 PLACE VALUE NOTATION

TEACHING IS NOT ONLY TEACHING IN THE MOMENT.

IT MUST BUILD ON WHAT STUDENTS BRING FROM THEIR PAST.

AND IT MUST PREPARE THEM FOR WHAT IS TO COME IN THEIR FUTURE.

WHAT DOES THIS IMPLY ABOUT THE TEACHING OF WHOLE NUMBERS?

. . . we assumed that the students' creation of a detailed and thorough conception of a real number, underlying which is the concept of quantity, is the purpose of this entire subject, from grade 1 to 10 . . . the teacher, relying on the knowledge previously acquired by the children, introduces number as a . . . representation of a general relationship of quantities, where one of the quantities is taken as a measure and is computing the other.

- Vasily Davydov

STORY LINE(S) OF THE NUMBER LINE

A Fundamental Mathematical Duality

Discrete	Continuous
Counting	Measurement
Whole numbers	Real numbers

Two early images of the number line

. . . • • • • • • • . . .

Cardinal/Ordinal World: The Number Queue

. . . ——|——|——|——|——|——|——|—— . . .

Measurement World: The Number Line

WHOLE NUMBERS ——> FRACTIONS

- A whole number is, conceptually, a mathematical object. "7"
- A whole number is the measure (cardinal) of a set.
- Addition/subtraction corresponds to composition/decomposition (set union)
- Multiplication to repeated addition (whole number rescaling), or to cartesian arrays.
- Whole numbers are named with our base-10 positional notation.
- Computational algorithms are anchored in this notation.
- Whole numbers are born in the cardinal/ordinal world.

- A fraction is, conceptually, a mathematical operator. "3/4 of"
- A fraction is the relative measure of two quantities.
- Addition/subtraction corresponds to composition/decomposition
- Multiplication to composition of operators, or to cartesian product. (The product is a different species of quantity.)
- Fractions are named with the fraction bar notation.
- Computational algorithms are anchored in this notation.
- Fractions are born in the worlds of (possibly continuous) measure.
- The cardinal world is one of these, though not typically seen this way.

THE HOME THAT THESE TWO POPULATIONS
(WHOLE NUMBERS AND FRACTIONS)
FINALLY COME TO HARMONIOUSLY COHABIT
IS THE CONTINUOUS REAL NUMBER LINE

WITH DAVYDOV, I BELIEVE THAT
BUILDING THIS HOME,
TO FACILITATE THIS RAPPROCHEMENT,
SHOULD BEGIN IN THE EARLIEST GRADES

HOW?

WHAT MATHEMATICAL THINKING DO CHILDREN FIRST BRING TO SCHOOL?

AND

HOW CAN THIS BE USED IN THE SCHOOL CURRICULUM?

BUTTERWORTH (AMONG OTHERS)

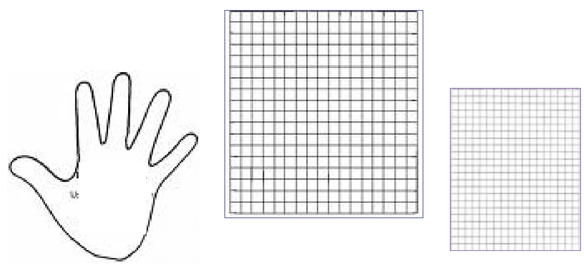
- "*God made the integers, all else is the work of man.*"
  - Leopold Kronecker
- It is now widely acknowledged that the typical human brain is endowed by evolution with a mechanism for representing and discriminating numbers . . . . when I talk about numbers I do not mean just our familiar symbols – counting words and 'Arabic' numerals, I include any representation of the number of items in a collection, more formally the cardinality of the set, including unnamed mental representations. Evidence comes from a variety of sources.

DOUGLAS CLEMENTS AND MICHELLE STEPHAN
(AMONG OTHERS)

Children's understanding of measurement has its roots in the preschool years. Preschool children know that continuous attributes such as mass, length, and weight exist, although they can not quantify or measure them accurately. Even 3-year-olds know that if they have some clay and then are given more clay, they have more than they did before. Preschoolers cannot reliably make judgments about which of two amounts of clay is more; they use perceptual cues such as which is longer. At age 4-5 years, however, most children can learn to overcome perceptual cues and make progress in reasoning about and measuring quantities.

Measurement is defined as assigning a number to a continuous quantity.

MEASURING AREA IN FIRST GRADE (D. BALL)



TWO NARRATIVES OF THE JOURNEY TO THE REAL NUMBER LINE:

- *CONSTRUCTION* (START WITH COUNTING)

AND

- *OCCUPATION* (START WITH MEASURE)
(in full experiential generality)

THE CONSTRUCTION NARRATIVE:

STARTING WITH WHOLE NUMBERS, WE PROGRESSIVELY INVENT MORE NUMBERS.

IN ALL BUT ONE STEP: WE INVENT NEW NUMBERS TO SOLVE PROBLEMS STATED, BUT NOT SOLVABLE, IN THE PREVIOUS SYSTEM.

HOWEVER, THE PASSAGE FROM RATIONAL NUMBERS TO REAL NUMBERS IS MORE SOPHISTICATED, AND REMAINS SOMEWHAT MYSTERIOUS.

CONSTRUCTION NARRATIVE OVERVIEW:
BUILD THE NUMBER LINE BY SUCCESSIVE ENLARGEMENT

Premise: Children come to school with some rudimentary skills of counting.

I. Whole numbers
II. Integers (Solve $a + x = b$) (Subtraction)
II'. Fractions (Solve $a \bullet x = b$) (Division; measurement)
III. Rational numbers (integrate II and II')
IV. A few naturally occurring irrational numbers: $\sqrt{2}$, π, (e)
V. Real numbers ("rationals plus irrationals" Conceptual gap)
VI. Complex numbers (Solve $x^2 + 1 = 0$)

A CURRICULUM

BASED ON THE CONSTRUCTION NARRATIVE

OFTEN FALLS SHORT OF

GIVING HIGH SCHOOL GRADUATES

A ROBUST CONCEPTUAL UNDERSTANDING

OF THE CONTINUOUS NUMBER LINE

HANS FREUDENTHAL

The device beyond praise that visualises magnitudes, and at the same time the natural numbers articulating them, is the number line, where initially only the natural numbers are individualised and named. In the didactics of secondary instruction the number line has been accepted, though it is often still imperfectly and inexpertly exploited

Quoted in Bartolini Bussi, *The Number Line*, in Theme 3 of ICMI Study 23 Proceedings

THE OCCUPATION NARRATIVE:

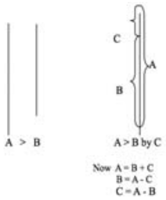
THE GEOMETRIC (NUMBER) LINE IS PRESENT FROM THE BEGINNING

THE NUMBERS PROGRESSIVELY TAKE UP RESIDENCE THERE
(VIA LINEAR MEASURE)

(DAVYDOV'S FOUNDATION)

QUANTITY (chez Davydov)

- Young children have a primordial sense of <u>quantity</u>, an attribute of physical objects (not only cardinals): length, area, volume, time, ... Without numerical associations.
- And of <u>addition</u> (composing and decomposing quantities, of the same species)
- Rough comparison of size ("Which is more?"), expressed as "A > B." And then infer that "A = B + C" for the "difference" C.
- Davydov develops in children such <u>algebraic relations</u>, involving "pre-numerical" quantities, and hence <u>with no numerical calculation</u>.
- Pre-cursor of algebraic thinking. And it imparts the correct sense of the "=" sign.
- He develops these ideas in first grade, prior to the introduction of (whole) numbers, in a measurement context



MEASURE AND NUMBER

- A quantity has no intrinsically attached number.
- Rather, given two quantities, A and U, then, taking U as a "unit," the number we attach to A is, "How much (or many) of U is needed to constitute A?" Thus <u>a number is a ratio of two quantities.</u>
- To understand a numerical quantity, it is necessary to specify, or know, the unit. And, for a given species of quantity, different units may be chosen: (feet, inches, meters – for length), (quarts, pints, liters – for liquid volume), etc.
- <u>To numerically simplify a sum</u> of two numerical quantities they must be of the same species and expressed with the same unit. ("Can't add applies and oranges.")
- That is why, in place value algorithms for addition, we vertically align the digits with the same place value, i.e. with the same base-10 units.
- That is why, in adding fractions, we seek common denominators ("unit fractions").
- In principle, numerical quantities manifest <u>the full continuum of (positive) real numbers.</u>
- <u>Whole numbers arise, in every measure regime</u>, when a quantity is composed exactly of a set of copies of the unit.
- <u>This is how to comprehend whole numbers in the general measure context, not simply cardinal counting.</u> (In the cardinal world, the unit is the one element set, and each set is composed of a set of copies of this unit.)

DAVYDOV'S FUNDAMENTAL PROBLEM OF MEASURE: LIFT OFF

- **Given a quantity A, reproduce A in a different place and time.**
- <u>With children</u>: A strip of tape, A, is on a table. In the next room is a roll of tape. <u>Task</u>: Cut off a piece of that tape exactly the length of A. You are not allowed to move A.
- <u>Different approaches</u>:

0. Make a guess, from a remembered image. (Very inexact.)
1. Cut off a piece of string the length of A. (<u>Need for a mediating quantity</u>.)
2. Suppose you have a stick of wood, longer than A. Mark it at the length of A.
3. Suppose you have a piece of wood shorter than A. Count off lengths of the piece to *measure* A. (<u>The child constructs the idea of measurement, and engages the concept of unit.</u>)

(**Brousseau**: didactical situation. **Harel**: Intellectual necessity.)

- <u>This leads the learner to the concepts of measure and of unit.</u>
- Imagine this experiment with cardinal instead of linear measure. Several conceptual and cognitive steps would be missing. Approach 0 would suffice.

COORDINATIZING THE GEOMETRIC LINE

("Descartes in dimension one")

- The "geometric line" is the environment of linear measure.
- A helpful metaphor is an (indefinitely long) string (<u>flexible, but inelastic</u>).
- Then linear quantities, like your height, or your hat size, can be represented, as quantities, by a segment of string (no numerical strings attached).
- The geometric line is coordinatized with numbers by <u>choice of an ordered pair of points, that we call 0 and 1</u>. Then we take the interval [0, 1] as the unit of linear measure. Moreover, the direction from 0 to 1 is taken as the positive orientation of the line. (The line has an intrinsic "linear structure," arising from "betweenness," allowing for <u>two possible "linear orders</u>.")
- Our convention is to depict the line horizontally, and to take (left –> right) as positive.
- A whole number N is then placed on the line by concatenating, to the right, N copies of the unit, starting at the origin, and placing N at the right endpoint.
- This achieves the merger of the counting model of whole numbers with the line model based on (continuous) linear measure.
- And later, fractions are placed on the line by a similar process, using a "subunit" specified by the denominator.

WHAT HAS BEEN ACHIEVED BY THIS APPROACH?

- The part-whole introduction of fractions IS a measurement approach, the *whole* being the unit of measure.
- Though counting is also a measurement context, that point of view is not emphasized, since there is a natural default choice of unit (the 1-element set), and so the very concept of unit, and its possible variability, does not enter conscious reflection or discussion.
- That is why fractions can seem such a different species from whole numbers. In the measure approach, <u>whole numbers are present in every measurement regime.</u>
- Moreover, placement of whole numbers on the number line requires appeal to (continuous) linear measure.
- Finally, <u>the geometric line is present from the beginning, and the progressive enlargements of our number world simply supplies names to more and more of the (already present) points on the line</u>.
- This is the "occupation narrative" of the real number line.

APPEARANCE OF THESE IDEAS

IN THE NEW U. S.

COMMON CORE STADARDS

EARLY MEASUREMENT IN THE U.S. COMMON CORE STANDARDS

- Kindergarten: Describe and compare measurable attributes.
- Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object. (Quantities)
- Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

- Grade 1: Measure lengths indirectly and by iterating length units.
- Order three objects by length; compare the lengths of two objects indirectly by using a third object. (Order in linear measure)
- Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps. (Whole number linear measure)*

GRADE 2:

- Measure and estimate lengths in standard units.
- Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes. (Units)
- Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen. (Different units)
- Estimate lengths using units of inches, feet, centimeters, and meters.
- Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.
- Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram. (Whole numbers on the number line)

GRADE 3: PLACING FRACTIONS ON THE NUMBER LINE

- Understand a fraction as a number on the number line; represent fractions on a number line diagram.
  - Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
  - Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

SOME HISTORY (see Bartolini-Bussi, Theme 3, for a fuller discussion)

- Vasily Davydov (1930 – 1988), Vygotzkian psychologist and educator, in the Soviet Union.
- With colleagues, in the 1960s, he developed a curriculum starting with quantity (of real objects) and measure.
- Adaptations of the Davydov curriculum have been implemented in the U. S., with some claims of success.
  - Hannah Slovin & Barbara Dougherty, U. Hawaii, "Measure Up" (2004)
  - Jean Schmittau, SUNY Binghamton, (2005)
  - Peter Moxhay, Portland, Maine Public Schools, (2008)
- Many of these ideas are present in the NCTM and Common Core Standards, but are not yet enacted in most instruction.

PART I
QUANTITIES, NUMBERS, AND THE NUMBER LINE

PART II
NUMBER NAMES
BASE-10 PLACE VALUE NOTATION

The decimal system is so widely used because it is wonderful. It is a tool of remarkable sophistication and efficiency. It provides a unique way to represent each whole number, no matter how large, in compact form. Moreover, it supports computation with efficient and learnable algorithms for the arithmetic operations. It also makes comparison and estimation easy. Thus, it supports all our efforts of quantitative reasoning. In schools throughout the world, learning whole number arithmetic means studying the decimal system.

- From **Roger Howe**, *The most important thing for your child to learn about arithmetic*, ICMI Study 23, Theme 1.

- Howe often quotes another champion of place value:

The greatest calamity in the history of science was the failure of Archimedes to invent positional notation. - **Carl Friedrich Gauss**

RECEIVED OR CONSTRUCTED KNOWLEDGE?

- We have seen many ways that children can be taught to understand and use place value, as received cultural knowledge.
- From a social constructivist perspective, can we design a didactical situation (Brousseau) the creates the intellectual need (Harel) to construct a positional system of numeration?
- I want to show you one such design. It is enacted with pre-service teachers (PSTs), in a simulation that might be enacted with children.

THE DESIGN: THE BEGINNING

- Half the class sits on the floor, with the teacher, pretending to be young children. The other half observes the teacher questioning and interactions.
- The teacher spills a bin of MANY sticks on the floor, and asks how many?
- Guesses. . . . How could we find out? . . . Count them.
- Collective counting starts, but each PST soon gets so many sticks it is hard to keep track. . . . Bundle them.
- How many in a bundle? Each bundle should be the same amount, even for different PSTs. Need to assemble collections at the end.
- Discuss various choices (2, 5, 10, …) Settle on 10, but note that a choice was needed, and made. (Other bases.)

HOW MANY STICKS DO YOU THINK THERE ARE?

NEXT STAGE:

- The sticks are so numerous that the same problem arises from having too many (10-)bundles to keep track. So make bundles of bundles – "super-bundles."
- How many bundles in a super-bundle? Again a choice. Convenient to make it 10 again.
- Finally, each PST has a small number (< 10) of loose sticks, bundles, and super-bundles.
- When all are gathered together, need to build new bundles, super-bundles, and now also "mega-bundles" (made of 10 super-bundles)
- In the end, the cardinal can be encoded by 4 numbers < 10: the numbers of loose sticks, bundles, super-bundles, and mega-bundles. (Notational compression)

FROM A LARGE SET TO A GROUPING STRUCTURE
REPRESENTATIONAL COMPRESSION

Next step: Positional notation

NEXT STEPS: TOWARD POSITIONAL NOTATION

- How do we further compress

  2 mega-bundles + 4 super-bundles + 7 bundles + 6 sticks ?

  just using the numbers 2, 4, 7, and 6.
- How will we know what "unit size" each number is attached to?
- Positional notation.
- What if there were no super-bundles? Invent 0!

THANK YOU

hybass@umich.edu