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Optimal Property Management Strategies

Peter F. Colwell
Department of Finance, College of Commerce, University of Illinois at Urbana-Champaign, Champaign, Illinois 61820, USA or pcolwell@uiuc.edu

Yuehchuan Kung
GMAC – RFC, 8400 Normandale Lake Blvd. Suite 600, Minneapolis, MN 55437, USA or yuehchuan.kung@gmacrfc.com

Tyler T. Yang
Freddie Mac, 8200 Jones Branch Drive., McLean,VA 22209, USA or tyler_yang@freddiemac.com

This paper examines the optimal operation strategies for income properties. Specifically, the rental rate and the operating expense should be set at levels to maximize the return on investment. The results suggest that for a given demand curve of a specific rental property, there exist optimal levels of the income ratio, the operating expense ratio, and the vacancy rate. With a Cobb-Douglas demand curve, we derived closed form solutions of these optimal ratios for a given income property. The relevant local comparative statics of these ratios also are derived. These comparative statics also provide insight into the optimal building size and optimal rehabilitation decisions. An empirical case study was conducted to demonstrate how the model can be applied in real life situations.

Keywords
Rental Property; Vacancy Rate; Operating Strategy; Profit Optimization

Introduction

The purpose of this paper is to analyze operating strategies for income properties. Two of the most important decisions for managers to make are the amount of rent and operating expenses that should be charged or be spent on the rental property. These issues were rarely studied in the academic literature. In this paper, we attack this problem by assuming that the manager's objective is to maximize the net present value (*NPV*) of the investment.

Colwell (1991) indicated that, under a given market condition, a higher vacancy rate might actually be preferable to a lower vacancy rate. He used a downward sloping curve between occupancy rate and gross rental rate to illustrate that, within a relevant range, per-unit rent must fall if occupancy is to rise. A completely occupied building may not provide the maximum possible net operating income (*NOI*) to the property owner. Colwell concluded that in maximizing value, achieving a precise balance in income and expenses might be more important than reaching 100-percent occupancy.

Chinloy and Maribojoc (1998) used a portfolio of apartment buildings in Portland, Oregon, to test whether managers have flexibility to select strategies on expense (overhead, repairs, capital expenditures, taxes and insurance, and marketing)-rent combinations. They found a positive correlation between gross rents and expenses. However, the correlation coefficients between net rents and expenses are not always positive. *NOI* increases with an increase in the marketing expense, but decreases with an increase in other expenses. They contended that optimization at the margin is not always achieved. There is scope for increases at the margin in certain expense categories and reduction in others, though partly mitigated by the lumpiness of investments.

In the next section, we introduce the profit maximization decisions faced by an investor. Algebraic properties of the optimal operating ratios are also introduced. In Section 3, we use a Cobb-Douglas demand curve to demonstrate the maximization solution more precisely. Comparative statics of the closed-form solutions of the optimal strategies are derived in Section 4. From these comparatives, implications regarding the optimal building size and optimal rehabilitation strategy are also discussed. An empirical analysis is conducted and summarized in Section 5. The corresponding optimal operating ratios and their sensitivities confirm with the algebraic solutions. The last section provides conclusions and possible extensions.

The Optimization Framework

Cannaday and Yang (1995, 1996) discussed real estate investor's optimal financial decisions (*i.e.* the optimal interest rate-discount points combination and the optimal leverage ratio strategies) of income-producing properties. Both studies focus on income-producing properties, and are based on a discounted cash flow approach. In this paper, we adopt the identical approach. Instead of analyzing the financing decisions, we focus on the investment decisions. Specifically, we study the optimal operating strategy in terms of setting the levels of rent and operating expenses.

As in most other investment situations, a typical equity investor in the rental market tries to maximize the *NPV* from investment over a given investment horizon. For income producing properties, the income consists of the after tax cash flows (*ATCF*) and the after tax equity reversion (*ATER*), while the initial investment outlay consists of the price of the real estate and the associated transaction costs incurred in acquiring the property. The *ATCF* is the rental income less such associated expenses as expenses for operation and maintenance, mortgage debt service, and income taxes. The *ATER* is the future sales price less such associated expenses as transaction costs, mortgage repayment, and capital gains taxes.

To simplify the problem, we ignore tax effects and focus on the pre-tax *NPV*. Without capital gains taxes, the *ATER* would be equivalent to the before tax equity reversion (*BTER*). Meanwhile, without income taxes, the *ATCF* would be the same as the before tax cash flow (*BTCF*).

Without losing generality, we assume that the property is 100% equity financed[1]. When there are no mortgage expenses, the *BTCF* is the same as the *NOI*, which is defined as the effective gross income (*EGI*) less the operating expenses (*OE*). The *EGI* is the potential gross income (PGI) minus the vacancy and collection losses. If we further assume that the rent is the only revenue generated by the property, then the PGI can be computed as the per unit rent (R_t) multiplied by the number of available rental units (Q^s) - that is, the quantity of rental space within a particular property.

The measuring unit for rental space can be dwelling units for residential properties or square feet for non-residential properties. When there are no collection losses, the EGI is equal to $R\,Q^d$, where R is the rent (price) per unit of rental space and Q^d is the quantity of rental space demanded by potential renters. Let $C = \frac{OE}{Q^s}$ be the per unit operating expenses, or the cost incurred by the investor for each unit of available rental space. For a given physical property, higher C usually leads to higher quality, and is more attractive to potential renters.

[1] Modigliani and Miller demonstrated through their "proposition I" that in a perfect market, the capital structure is irrelevant to the value of a firm.

Collectively, the *NOI* can be written as $R\ Q^d - C\ Q^s$. However, Q^d, the quantity demanded, is subject to the physical constraint of Q^s, the physical space available for lease. If Q^d is smaller than Q^s, Q^d is occupied and the vacancy rate is $\frac{Q^s - Q^d}{Q^s}$. On the other hand, if Q^d is greater than Q^s, only Q^s space can be leased due to the physical constraint and the vacancy rate being zero. Therefore, the profit maximization problem is subject to the constraint: $Q^d \leq Q^s$.

An investor's objective is to maximize the net present value (NPV). Following the above simplifications, the NPV for an income property is the present value of the cash flows plus the present value of the equity reversion minus the initial investment.

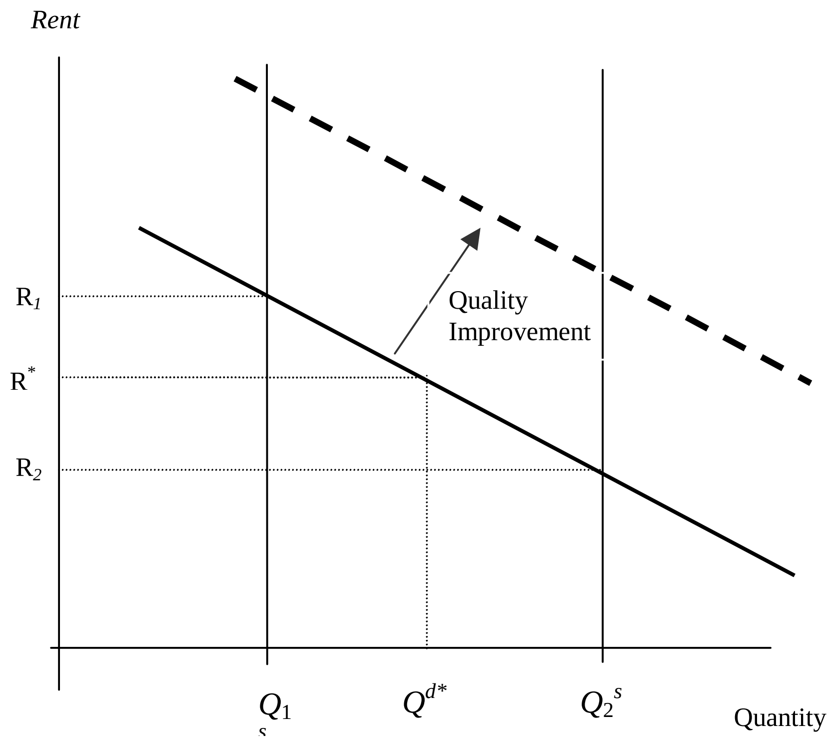
$$\mathrm{Max}_{\{Rt,Ct\}}\ \mathrm{NPV} = \sum_{t=1}^{T}\left(\frac{R_t Q_t^d - C_t Q_t^s}{(1+i)^t}\right) + \frac{P_T}{(1+i)^T} - P_0 \qquad (1)$$

s.t. $Q^d{}_t \leq Q^s{}_t$,

where i is the investor's required rate of return or cost of capital, which is determined exogenously, T is the expected number of periods before the property is re-sold, and P_0 is the price (cost) initially paid for the property.

In the short run, the supply curve is a vertical line or perfectly inelastic. On the other hand, the demand for rental space depends on the rent and the operating expenses, $Q^d{}_t[\ R_t, C_t\ ]$. On a regular price-quantity plane as shown in Figure 1, the change in operating expenses corresponds to a shift in the demand curve, while the change in rent corresponds to a move along the demand curve. Given a specific physical property, the quality of the housing services of each physical rental unit (apartment or house) increases with the discretional expenses landlords spend to maintain the property and provide extra amenities. Thus, holding rent constant, higher amenities usually lead to higher demand. As a result, the demand curve shifts out with the operating expenses. This is illustrated in Figure 1, when *OE* increases, and the demand curve shifts out from the solid curve to the dashed curve. On the other hand, holding operating expenses or the quality of a property constant, if rent level was very high, one would expect that decreasing the rent marginally would attract higher demand. This represents a move along the demand curve in Figure 1.

Figure 1 The Demand Curve of a Rental Property



These behaviors lead to the following properties of the demand curve with respect to *R* and *C*:

$$\begin{cases} \dfrac{\partial Q^{d}{}_{t}}{\partial R_t} < 0, \\ \dfrac{\partial Q^{d}{}_{t}}{\partial C_t} > 0. \end{cases} \tag{2}$$

Assuming that the demand function is independent of time, all the time subscript in the demand function can be dropped from Equation (1). Furthermore, assuming that the real estate transaction market is informationally efficient, real property must be sold at a price equal to the maximum present value of the future *NOI*'s[2]. According to Gordon's rule, the value of an investment is the future potential income discounted by the investor's required return. Every investor who is interested in purchasing the property will operate the property so as to maximize his *NPV* from the investment. When the market is informationally efficient, the winner of the bid for the property at time T must pay a price equal to the maximized present value of the *NOI* he is able to obtain from the property. Thus, the future sales price, P_T, should be the maximized present value of the *NOI* discounted by the cost of capital, i.

$$P_T = \text{Max}_{R,C} \sum_{t=T+1}^{\infty} \left(\frac{RQ^d - CQ^s}{(1+i)^t} \right) \tag{3}$$
$$\text{s.t. } Q^d \leq Q^s.$$

Substituting Equation (3) into equation (1), the objective function becomes:

$$\text{Max}_{R,C} \sum_{t=1}^{\infty} \left(\frac{RQ^d - CQ^s}{(1+i)^t} \right) - P_0 = \frac{RQ^d - CQ^s}{i} - P_0 \tag{4}$$
$$\text{s.t. } Q^d \leq Q^s.$$

Because P_0 is a fixed amount of sunk cost and i is exogenously determined by the investor's cost of capital, they are irrelevant to the maximization problem of Equation (4). Solving Equation (4) is equivalent to solving Equation (5):

$$\text{Max}_{R,C} \; R\,Q^d - C\,Q^s \tag{5}$$
$$\text{s.t. } Q^d - Q^s \leq 0.$$

Note that Equation (5) is nothing more than the maximization of a single period's *NOI*. Under the assumption of a time-independent demand curve, the multiple period model collapses into a single period condition. Investors would act as if they were myopic.

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[2] Evans (1991) discussed the meaning of market value and whether market or investment value represents "real" value. In a soft real estate market, there are not many owners willing to sell at these lower prices, so in effect, there are not two parties to the assumed transactions. As a result, one has liquidation values being presented by appraisers in the guise of market values. In such a case, the informationally efficient real estate transaction market assumption would no longer be valid.

Denote the optimal solutions as R^* and C^*, as they can be used in computing the optimal levels for the ratios commonly used in the real estate leasing industry. First, the optimal vacancy rate can be computed as:

$$V^* = \frac{Q^s - Q^{d*}}{Q^s} = 1 - \frac{Q^d[R^*, C^*]}{Q^s}. \tag{6}$$

In reality, if the demand for rental space is high enough, this rate can be brought down to zero. Figure 1 shows that having a zero vacancy rate may or may not be the optimal strategy. Suppose the unconstrained optimal quantity to be leased is Q^{d*} with R^*. If the supply is smaller than this Q^{d*}, such as Q_1^s in Figure 1, the constraint is binding. The manager can increase the rent to R_1 and still maintain a zero vacancy rate; yielding a higher *NOI*. On the other hand, if supply were greater than Q^{d*}, such as Q_2^s in Figure 1, then the optimal vacancy rate would be greater than zero, but the profitability also decreases. This is consistent with Colwell's (1991) finding. Of course, by lowering the rent to R_2, the manager could bring the vacancy rate down to zero.

Two other popular operating ratios referred to in the industry are the Income Ratio (*IR*) and the Operating Expense Ratio (*OER*). Their optimal levels under this framework are:

$$IR^* = \frac{NOI^*}{PGI^*} = \frac{R^* Q^d[R^*, C^*] - C^* Q^s}{R^* Q^s}, \text{ and} \tag{7}$$

$$OER^* = \frac{OE^*}{EGI^*} = \frac{C^* Q^s}{R^* Q^d[R^*, C^*]}. \tag{8}$$

Again, whether the result of Equation (8) is consistent with the rule of thumb is hard to determine. The example provided in the next section suggests that the rule of thumb fails to provide a unique operating strategy.

A Specific Solution

A Cobb-Douglas demand function is used to give a more precise sense of how the above optimal strategies work. The specific form of the demand curve we choose is:

$$Q^d = Q^0 - \alpha \frac{R^\beta}{C^\delta}, \tag{9}$$

where Q^0 is the total potential demand for rental space if no rent is required, α is a scalar that measures the effect of operating strategy on the quantity of rental space demanded,

$\beta = \varepsilon_R \frac{Q^d}{Q^0 - Q^d}$ is a measure of rent elasticity of demand (ε_R is the rent elasticity of demand),

$\delta = -\varepsilon_C \frac{Q^d}{Q^0 - Q^d}$ is a measure of operating expense elasticity of demand (ε_C is the operating expense elasticity of demand), and

$Q^0, \alpha, \beta, \delta > 0$.

This Cobb-Douglas demand curve is a flexible and reasonable functional form to capture the local behavior of the demand curve. The local concavity behavior is applicable to a wide range of demand curves that satisfy the properties in Equation (2). In particular, we have:

$$\begin{cases} \dfrac{\partial Q^d}{\partial R} = -\alpha\beta \dfrac{R^{\beta-1}}{C^\delta} < 0, \\ \dfrac{\partial Q^d}{\partial R} = \alpha\delta \dfrac{R^\beta}{C^{\delta+1}} > 0. \end{cases} \tag{10}$$

Substituting this demand function into Equation (5), we are able to solve for the optimal level of rent and operating expense, and to obtain insight into the validity of the rules of thumb. Because of the existence of the inequality constraint, the problem can be solved for two conditions: 1) the constraint is not binding; and 2) the constraint is binding.

Constraint not binding

When the constraint is not binding, the first order necessary conditions for the optimization problem are:

$$\begin{cases} \dfrac{\partial NOI}{\partial R} = Q^0 - \alpha(\beta+1)\dfrac{R^\beta}{C^\delta} = 0, and \\ \dfrac{\partial NOI}{\partial C} = \alpha\delta \dfrac{R^{\beta+1}}{C^{\delta+1}} - Q^s = 0. \end{cases} \tag{11}$$

Solving the simultaneous Equation (11), the optimal rent and optimal operating expense are found to be:

$$\begin{cases} R^* = \left(\dfrac{Q^{0^{1+\delta}} \delta^{\delta}}{\alpha Q^{s^{\delta}} (1+\beta)^{1+\delta}} \right)^{1/(\beta-\delta)}, \text{ and} \\ C^* = \left(\dfrac{Q^{0^{1+\beta}} \delta^{\beta}}{\alpha Q^{s^{\beta}} (1+\beta)^{1+\beta}} \right)^{1/(\beta-\delta)}. \end{cases} \tag{12}$$

To make sure that this solution is indeed the maximum instead of a minimum or a saddle point, we double-check the second order conditions. The three-second order partial derivatives for the *NOI* are:

$$\begin{cases} \dfrac{\partial^2 NOI}{\partial R^2} = -\alpha\beta(1+\beta)\dfrac{R^{\beta-1}}{C^{\delta}} < 0, \\ \dfrac{\partial^2 NOI}{\partial C^2} = -\alpha\delta(1+\delta)\dfrac{R^{\beta+1}}{C^{\delta+2}} < 0, and \\ \dfrac{\partial^2 NOI}{\partial R \partial C} = \alpha\delta(1+\beta)\dfrac{R^{\beta}}{C^{\delta+1}}. \end{cases} \tag{13}$$

The first two of these second order partial derivatives carry negative signs, and thereby guarantee that the solution set is not a minimum. To ensure that the solution is not a saddle point, the necessary condition for the solution to satisfy is :

$$\left(\frac{\partial^2 NOI}{\partial R^2}\right)\left(\frac{\partial^2 NOI}{\partial C^2}\right) - \left(\frac{\partial^2 NOI}{\partial C \partial C}\right)^2 = \beta - \delta > 0. \tag{14}$$

This result indicates that $\beta > \delta$ is the only additional requirement to ensure the Equation (12) solution set is the maximum.
Plugging Equation (12) into Equation (9), we find the optimal quantity demand to be:

$$Q^{d*} = Q^0 \quad \frac{\beta}{1+\beta} > 0. \tag{15}$$

Recall that the solution set we computed above is for the case in which the constraint is not binding. For this solution to be an interior solution, Q^{d*}

must be smaller than the quantity supplied. This is the same as requiring $Q^S \geq \frac{\beta}{1+\beta} Q^0$. The optimal *NOI* the landlord obtains by using the optimal strategy (R^*,C^*) is found to be:

$$NOI^* = (\beta-\delta)\left(\frac{Q^{0^{1+\beta}}\delta^{\delta}}{\alpha Q^{S^{\delta}}(1+\beta)^{1+\beta}}\right)^{1/(\beta-\delta)}. \tag{16}$$

Since $\beta > \delta$, the NOI^* is always greater than zero. This result ensures that the solution set is not dominated by the trivial solution that $R = C = 0$. Substituting the solution set into Equations (6), (7), and (8), we find the operating ratios indicated by the strategy of maximizing the net operating income:

$$\begin{cases} V^* = 1-\dfrac{\beta Q^0}{(1+\beta)Q^S}, \\ IR^* = \dfrac{(\beta-\delta)Q^0}{(1+\beta)Q^S}, \text{ and} \\ OER^* = \dfrac{\delta}{\beta}. \end{cases} \tag{17}$$

These optimal ratios demonstrate several interesting points. First, V^*, the optimal vacancy rate is always smaller than or equal to 100 percent because the second term of the first line of Equation (17) is always positive. On the other hand, because the physical constraint is not binding, the quantity demanded must be smaller than Q^S. Therefore, for the solution set to be the maximum with the constraint not binding, the quantity of space supplied must be greater than the quantity demanded. This criterion prevents the second term in the first line of Equation (17) from being greater than 1. That is, the V^* is greater than 0. Otherwise, it belongs to the case where the constraint is binding. For a given market condition, a larger building is more likely to realize an interior solution. Satisfying this criterion is equivalent to saying that the optimal vacancy rate is strictly greater than zero. A zero vacancy rate fails to provide the maximum possible profit. By merely increasing the rent level, the landlord can increase the *NOI*. When the operating expense is adjusted simultaneously, the *NOI* can be brought to an even higher level. Second, the income ratio is also between 0 and 1. The relationship $\beta > \delta$, which we obtained from Equation (14), guarantees this income ratio to be positive. Also, since $Q^s > \frac{\beta Q^0}{(1+\beta)} > \frac{(\beta-\delta)Q^0}{(1+\beta)}$, the IR^* is always less than 1. Finally, the operating expense ratio is always between 0 and 1. It is obvious that this ratio can never be negative, and given that $\beta > \delta$, the OER^* is always smaller than 1.

Constraint Binding

If the optimal demand quantity obtained in Equation (15) is greater than the space available, then the physical constraint becomes binding, and we have $Q^d = Q^s$. Under such circumstances, the optimal operating expense can be written as a function of the optimal rent. That is,

$$C = \left(\frac{\alpha R^\beta}{Q^0 - Q^s}\right)^{1/\delta}. \tag{18}$$

Substituting Equation (18) into the objective function Equation (5), the problem is simplified to:

$$\mathrm{Max}_R \; RQ^s - Q^s\left(\frac{\alpha R^\beta}{Q^0 - Q^s}\right)^{1/\delta}. \tag{19}$$

The optimal solution of this objective function is:

$$\begin{cases} R^{**} = \left(\dfrac{(Q^0 - Q^s)\delta^\delta}{\alpha\beta^\delta}\right)^{1/(\beta-\delta)} \\ C^{**} = \left(\dfrac{(Q^0 - Q^s)\delta^\beta}{\alpha\beta^\beta}\right)^{1/(\beta-\delta)} \end{cases}. \tag{20}$$

Both R^{**} and C^{**} are guaranteed to be positive. If the constraint is binding, the potential demand, Q^0, must be greater than the space supplied.

Substituting Equation (19) into the objective function, we obtained the optimal *NOI*

$$NOI^{**} = (R^{**} - C^{**})\, Q^s = Q^s\left(\frac{(Q^0 - Q^s)\delta^\delta}{\alpha\beta^\beta}\left(\beta^{\beta-\delta} - \delta^{\beta-\delta}\right)\right)^{1/(\beta-\delta)}. \tag{21}$$

This NOI^{**} is greater than zero because β is greater than δ. Therefore, Equation (21) guarantees the existence of a non-trivial solution. Furthermore, since the constraint is binding, increasing the building size marginally implies an increase in *NOI*. The positive signs on the comparative statics introduced in the next section confirm this result.

Since the constraint is binding, we know that the space demanded, given the (R^{**}, C^{**}) strategy, equals the space supplied. That is, the optimal vacancy rate, V^{**}, is zero. In other words, when the demand for rental space is high, or $Q^0 >> Q^S$, maximizing *NPV* can be achieved by minimizing the vacancy rate.

If the constraint were binding, the quantity demanded would be equal to the quantity supplied. This condition simplifies the income ratio to be $1 - \frac{C}{R}$ and the operating expense ratio to be $\frac{C}{R}$. Substituting Equation (20) into *R* and *C* above, we get:

$$\begin{cases} IR^{**} = 1 - \dfrac{\delta}{\beta}, and \\ OER^{**} = \dfrac{\delta}{\beta}. \end{cases} \tag{22}$$

A constrained condition can be used to explain markets with rent control. In those markets, the rent elasticity is less than one, meaning the landlord can increase *NOI* by increase rent. This additional constraint leads to a zero vacancy rate. In order to increase *NOI*, the operating policy is to lower the operating expenses to the minimum level. As a result, the unit of housing services provided by each rental unit decreases. The demand curve shifts down. Landlords will continue this as long as it does not violate the safety codes.

To summarize, the objective of maximizing the investor's *NPV* does provide a unique optimal operating strategy for the given demand function. The physical constraint is likely to be binding when Q^S is much smaller than the total potential demand for space. The optimal strategy derived in this section can help property managers achieve the highest return on investments.

Comparative statics

In addition to the optimal operating strategy, investors may also be interested in the impacts of changes in market conditions (*i.e.* changes in Q^0, Q^S, $\alpha, \beta,$ and δ) on the profit maximization rent/expense combinations across sub-markets or over time. These impacts are analyzed by the comparative statics. Table 1 presents the signs of all the comparative statics for the *NPV* maximizing solution. Table 1 provides suggestions for adjustments in the levels of rent and operating expenses that should be undertaken with respect to the changes in the market parameters. Given the adjustment in operating strategy, we can also read the direction of the change in the *NOI* and operating ratios from Table 1. A positive sign indicates that the optimal level of the variable should increase with an increase in the market parameter. A negative sign indicates the reverse condition. A question mark implies that the movement in the optimal level of the variable can be either up or down, depending on the current market condition.

Table 1Results of the comparative statics

1(a) Constraint Not Binding

	∂R^*	∂C^*	∂NOI^*	∂V^*	∂IR^*	∂OER^*
∂Q^0	+	+	+	-	+	0
∂Q^S	-	-	-	+	-	0
$\partial \alpha$	-	-	-	0	0	0
$\partial \beta$	?	?	?	-	+	-
$\partial \delta$	?	?	?	0	-	+

1(b) Constraint Binding

	∂OER^{**}	∂R^{**}	∂C^{**}	∂NOI^{**}	∂V^{**}	∂IR^{**}
∂Q^0	0	+	+	+	0	0
∂Q^S	0	-	-	?	0	0
$\partial \alpha$	0	-	-	-	0	0
$\partial \beta$	-	?	?	?	0	+
$\partial \delta$	+	?	?	?	0	-

* Proofs of these comparative statics are shown in the Appendix.

Some of the comparative statics provide particularly interesting implications. Specifically, the impacts of the changes in the potential demand (Q^0) and the quantity supplied (or building size, Q^S) on the levels of rent and operating expenses provide information about the adjustment of the manager's operating strategy. Meanwhile, the impact of the change in Q^S on the optimal

NOI provides some insight into the optimal development and rehabilitation strategies.

First, a change in the quantity supplied has the same impact on the optimal levels of rent and operating expense regardless of whether the constraint is binding. If there is an increase in the quantity of the space available for lease, then the manager should lower the rent and cut the operating expense in order to achieve a new optimal *NOI*. It may sound strange that the physical size of the real property can change. Several conditions could induce a change in Q^S. One example would be if an existing building were torn down and replaced by a larger or smaller building. Another would be the case in which a new building was acquired by the management team and as a result, the quantity of total space supply controlled by the same manager was increased. Yet another case might be the conversion of owner-occupied space to renter-occupied space. In accordance with the change in the amount of space under control, the manager should adjust his operating strategy to achieve the new optimal *NOI*.

Second, an increase in the number of potential renters implies that the manager should raise the rent and operating expenses regardless of whether the constraint is binding or not. In either case, the optimal NOI increases. It is important for the manager to determine which condition he is facing in order to make the best adjustment.

Finally, while a greater Q^S implies a lower *NOI* when the constraint is not binding, the change in the optimal *NOI* with respect to the change in Q^S when the constraint is binding is indeterminate. The negative sign of $\frac{\partial NOI^*}{\partial Q^s}$ in Table 1(a) shows that a smaller building implies a higher *NOI*. Since our demand function does not depend on the Q^S, a decrease in Q^S decreases total costs while leaving total revenue unchanged. As a result, the NOI would be higher under such a circumstance. This result suggests that when developing a new rental building, the investor should make the building as small as possible. However, as the building size decreases, it is more likely that the physical constraint becomes binding. If the physical constraint holds, we should focus on the result provided in Table 1(b). The sign of $\frac{\partial NOI^{**}}{\partial Q^s}$ depends on the amount of space available. Specifically, the sign is negative if and only if Q^S is greater than $\frac{Q^0(\beta-\delta)}{1+(\beta-\delta)}$, and positive if and only if Q^S is less than $\frac{Q^0(\beta-\delta)}{1+(\beta-\delta)}$. We can denote the critical size as Q^*. This result reveals that the *NOI* will increase continuously with a decrease in Q^S until size Q^* is reached. When this particular size is reached, any further decrease in the building size will cause a decrease in the *NOI*. Thus, Q^* provides an optimal building size for the given market conditions. Recall that the quantity demanded under the optimal strategy when the constraint is not binding is $\frac{Q^0\beta}{1+\beta}$. This quantity is greater than Q^*. Therefore, we cannot derive the optimal building size by simply using $Q^d(R^*,C^*)$ as if the constraint were not binding. A developer considering building a new income property in the community should construct a building of this particular size. Also, when the investor is considering a rehabilitation project, he should try to adjust the existing building size toward this Q^*. If the increase in the present value of the *NOI* is greater than the cost of rehabilitation (by adding, partially tearing down, or totally rebuilding), then it is recommended that the investor do so.

The rest of the comparative statics in Table 1 also reveal some information about the effect of a change in market conditions. As the results are not directly applicable in property management, we will not discuss them in detail.

Empirical Analysis

The properties studied in this paper were apartment facilities in selected California communities. They range in size from ten units to complexes of over five hundred units. We examined all multi-family residential properties that were sold in three California counties between January 1993 and December 1995. Data about the transactions were obtained from COMPS, Inc., a subsidiary of the TRW organization. The transactions were classified according to their locations within Orange County, San Diego County, northern Los Angeles County, and western Los Angeles County. From the analysis, we excluded transactions with missing information and those involving properties with fewer than sixteen rental units. We thus obtained a working data set containing 467 transactions.

We used the Cobb-Douglas demand curve specified in Equation (9) with additional residential rental property characteristics controlling for physical quality deviations and size differences among properties. These variables include the age of the property (AGE in years), number of parking spaces per rental unit (PARKING), sales price per unit (U_PRICE), land size per unit (L_SIZE), total number of units in the property (UNITS), building size per unit (B_SIZE), and the average number of bedrooms (AVGROOM). Table 2 shows the summary statistics of variables.

Table 2 Summary Statistics of the Final Sample

Variable	N	Mean	Std Dev	Max	Min	Median
Q^d	467	30	33.39	316	8	21
R	467	7042	2084	23232	389	6805
C	467	1359	653.25	4057	44	1334
B_Size	467	769	228.86	1756	118	771
Age	467	23	16.78	94	1	22
Parking	467	1.25	0.59	2.59	0	1.21
Units	467	42	45.99	472	16	28
AVGROOM	467	1.54	0.69	3.8	0	1.55
L_SIZE	467	2120	14053	232143	153	886
U_Price	467	388199	21957	195000	2324	33553

The specific model we estimated takes on the following form:

$$Q^d = -\alpha \frac{R^\beta}{C^\delta} + \gamma_1 \text{ B\_SIZE} + \gamma_2 \text{ PARKING} + \gamma_3 \text{ UNITS} + \gamma_4 \text{ U\_PRICE} + \varepsilon. \tag{23}$$

Obviously, this is a non-linear function. A non-linear regression is applied to obtain the maximum likelihood estimates (MLE)[3]. Table 3 shows the results of this non-linear regression with only the significant independent variables. Except for α, all parameter estimates are significant at the 95 percent level. When we fixed the β and δ, the model becomes a linear regression, which has a R^2 equal to 0.95. With the estimated $\hat{\beta} > \hat{\delta}$, we know that a maximum *NOI* exists, and can be calculated by substituting these parameters into Equation (12).

[3] Different initial values of the parameters were tried to verify this as a global (vs. a local) solution.

Table 3 Final Model Results of the Non-Linear Regression

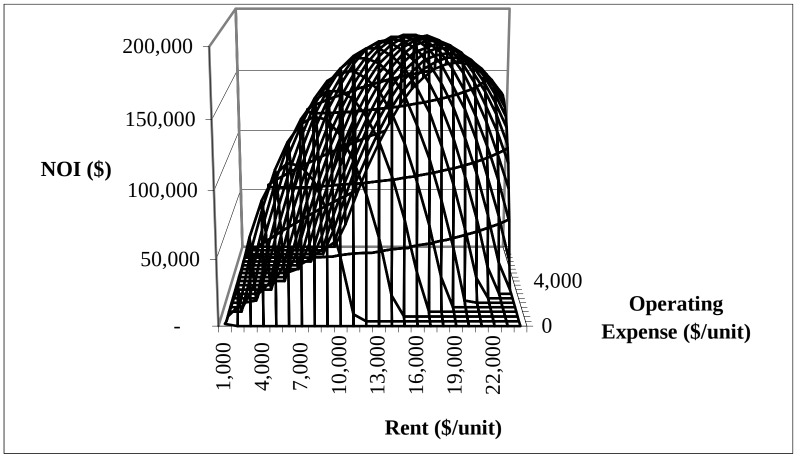
Independent Variable	Coefficient	p-Value
α	0.018339	0.2358
β	1.173158	<.0001
δ	0.502723	<.0001
B_SIZE	0.008307	0.0002
PARKING	1.629869	0.0107
UNITS	0.742138	<.0001
U_PRICE	0.000177	<.0001
Summary Statistics		
R^2	95%	
Number of Observations	467	

We used a typical property in the sample to illustrate the implications of these results. The typical property has the following characteristics: B_SIZE = 770, PARKING = 1.25, UNITS = 42, and U_PRICE = $38,900. Figure 2 illustrates the sensitivity of the *NOI* with respect to the *R* and *C* with the estimated model. For a typical rental property in our sample, the optimal strategy for rent and operating expense per unit are (R^* , C^*) = ($13,515, $3,461). That is, the landlord should spend $3,461 to maintain each rental unit, and the rental price should be $13,515 per unit. For this solution to be an interior solution, Q^{d*} must be smaller than the available quantity, 42 units. The quantity demanded is computed by substituting the estimated parameters into Equation (9). This Q^{d*} turns out to be 25 units, which is smaller than 42 units, the size of the building. As a result, the physical constraint is not binding. The optimal *NOI* is found to be $193,831 per period. We found the operating ratios indicated by the strategy of maximizing the net operating income:

$$\begin{cases} V^* = 40.25\%, \\ IR^* = 34.15\%, and \\ OER^* = 42.85\%. \end{cases}$$

We then compared the optimal R^*, C^*, V^*, IR^* and OER^* with the actual data on the individual property level. From Table 4, we found that the actual operating expenses and the actual rents in most properties were lower than the optimal solutions suggested by the model. Landlords could increase the NOI by increasing the quality of the property by putting in more operating expenses and simultaneously charging higher rents. Table 5 shows that the actual income ratios tend to be too high, and the actual operating expense ratios tend to be too low. Investors could increase profitability by increasing operating expenses more relative to the amount of increases in rent.

Figure 2 Net Operating Income of a Typical Apartment in Southern California



$$NOI = g\left(R, C \mid \mathrm{B\_SIZE}, \mathrm{PARKING}, \mathrm{UNITS}, \mathrm{U\_PRICE}\right)$$

B_SIZE = 770, PARKING = 1.25, UNITS = 42, and U_PRICE = 38,900.

Table 4 Comparison of R^*, C^* with R, C – Number of Properties

	1.2 C* < C	$0.8\ C^* < C < 1.2\ C^*$	$C < 0.8\ C^*$
$1.2\ R^* < R$	2	2	0
$0.8\ R^* < R < 1.2\ R^*$	5	40	69
$R < 0.8\ R^*$	2	16	331

R and C are the actual rent and operating expenses of the property. R* an C* are the optimal rent and operating expenses suggested by the model.

Table 5 Comparison of Operation Ratios – Number of Properties

	$1.3\ X^* < X$	$0.7\ X^* < X < 1.3\ X^*$	$X < 0.7\ X^*$
Vacancy Rate	101	174	192
Income Ratio	315	146	6
Operating Expense Ratio	9	26	432

X represents the actual vacancy rate, income ratio, or operating expense ratio realized from the property. X^* represents the optimal level suggested by the model.

One should notice that the sample represents a recession market. Southern California suffered from a prolonged recession in the early 1990's. The vacancy rate during this time period tends to be high. Finding ways to attract tenants is a very challenging task. The data observed in this empirical study reflects this condition. However, our model suggests that there is significant room for most property managers to improve their performance by resetting their pricing strategies. Sometimes, higher income ratios and lower operating expense ratios may not lead to maximum profitability.

Conclusions

In this paper, we showed that if the demand curve is time independent, then investors in rental properties are expected to behave as if they are myopic. That is, the strategy that maximizes the net present value of investment is identical to maximizing a single period's net operating income.

With a given demand curve, one can algebraically derive the operating strategies in terms of setting the levels of rent and operating expenses. We used a Cobb-Douglas demand curve to demonstrate the process and to study the properties of such optimal strategies. The impact of changes in market conditions on *NOI* over time or across sub-markets care were also revealed by comparative statics. These comparative statics also provided insights into the adjustment that one can make corresponding to the new environment to be maintained at the optimal position. These results also implied the ideal building size under a specific market. Rehabilitation could be optimal as the increase in the present value of net operating income exceeds the costs of adjusting the building size.

Cross-sectional multi-family transaction data from southern California was used to empirically estimate a Cobb-Douglas demand function. Based on the estimated parameters, we found that the actual operating expense and the actual rent tend to be lower than the optimal levels suggested by the model. On the other hand, the income ratio tends to be too high and the operating expense ratio tends to be too low. This suggests that when setting operating strategies, property managers should not simply focus on one or two ratios. Sensitivity of the demand (quantity) associated with the rent (price) and operating expense (quality) are just as important.

This model can also be used when the demand curve is estimated by time series data. With proprietary historical performance data, a property manager would be able to determine the optimal operating strategy. This is particularly valuable to managers of short-term lease properties, such as hotels.

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Appendix: Proof of the Comparative statics

Constraint Not Binding

1) Change of Q^0 :

$$\frac{\partial R^*}{\partial Q^0} = \mathbf{A}\ \frac{(1+\beta)(1+\delta)Q^s}{(\beta-\delta)Q^0} > 0,$$

$$\frac{\partial C^*}{\partial Q^0} = \mathbf{A}\ \frac{(1+\beta)\delta}{(\beta-\delta)} > 0,$$

$$\frac{\partial NOI^*}{\partial Q^0} = \mathbf{A}\ (1+\beta)Q^s > 0,$$

$$\frac{\partial V^*}{\partial Q^0} = -\ \frac{\beta}{(1+\beta)Q^s} < 0,$$

$$\frac{\partial IR^*}{\partial Q^0} = \frac{\beta-\delta}{(1+\beta)Q^s} > 0,$$

$$\frac{\partial OER^*}{\partial Q^0} = 0,$$

where $\mathbf{A} = \left(\dfrac{\delta^{\delta} Q^{0^{1+\delta}}}{\alpha(1+\beta)^{1+\beta} Q^{s^{\beta}}}\right)^{1/(\beta-\delta)} > 0.$

2) Change of Q^s

$$\frac{\partial R^*}{\partial Q^s} = -\ \mathbf{A}\ \frac{(1+\beta)\delta}{\beta-\delta} < 0,$$

$$\frac{\partial C^*}{\partial Q^s} = -\ \mathbf{A}\ \frac{\beta\delta Q^0}{(\beta-\delta)Q^s} < 0,$$

$$\frac{\partial NOI^*}{\partial Q^s} = -\ \mathbf{A}\ \delta\ Q^0 < 0,$$

$$\frac{\partial V^*}{\partial Q^s} = \frac{\beta Q^0}{(1+\beta)Q^{s^2}} > 0,$$

$$\frac{\partial IR^*}{\partial Q^s} = -\ \frac{(\beta-\delta)Q^0}{(1+\beta)Q^{s^2}} < 0, \text{ and}$$

$$\frac{\partial OER^*}{\partial Q^s} = 0.$$

3) Change of α

$$\frac{\partial R^*}{\partial \alpha} = -\,\mathbf{A}\ \frac{(1+\beta)Q^s}{(\beta-\delta)\alpha} < 0,$$

$$\frac{\partial C^*}{\partial \alpha} = -\,\mathbf{A}\ \frac{\delta Q^0}{(\beta-\delta)\alpha} < 0,$$

$$\frac{\partial NOI^*}{\partial \alpha} = -\,\mathbf{A}\ \frac{Q^0 Q^s}{\alpha} < 0,$$

$$\frac{\partial V^*}{\partial \alpha} = 0,$$

$$\frac{\partial IR^*}{\partial \alpha} = 0, \text{ and}$$

$$\frac{\partial OER^*}{\partial \alpha} = 0.$$

4) Change of β

$$\frac{\partial R^*}{\partial \beta} = \mathbf{A}\ Q^S \left(\frac{1+\beta}{(\beta-\delta)^2}\mathbf{D}+1\right),$$

$$\frac{\partial C^*}{\partial \beta} = \mathbf{A}\ \frac{\delta Q^0}{(\beta-\delta)^2}\,\mathbf{D},$$

$$\frac{\partial NOI^*}{\partial \beta} = \mathbf{A}\ Q^0\ Q^S \left(\frac{\mathbf{D}}{\beta-\delta}+2\right),$$

$$\frac{\partial V^*}{\partial \beta} = -\ \frac{Q^0}{(1+\beta)^2 Q^s} < 0,$$

$$\frac{\partial IR^*}{\partial \beta} = \frac{(1+\delta)Q^0}{(1+\beta)^2 Q^s} > 0, \text{ and}$$

$$\frac{\partial OER^*}{\partial \beta} = -\ \frac{\delta}{\beta^2} < 0,$$

where $\mathbf{D} = \log(\alpha) + (1 + \delta) \log(1 + \beta) - \delta \log(\delta) - (1 + \delta) \log(Q^0) + \delta \log(Q^S) - (\beta - \delta)$

5) Change of δ

$$\frac{\partial R^*}{\partial \delta} = - \mathbf{A} \quad \frac{(1+\beta)Q^s}{(\beta-\delta)^2}\,\mathbf{E},$$

$$\frac{\partial C^*}{\partial \delta} = - \mathbf{A} \;\; Q^0 \left(\frac{\delta}{(\beta-\delta)^2}\mathbf{E}-1\right),$$

$$\frac{\partial NOI^*}{\partial \delta} = - A \;\; Q^0\, Q^S \left(\frac{E}{(\beta-\delta)}+1\right),$$

$$\frac{\partial V^*}{\partial \delta} = 0,$$

$$\frac{\partial IR^*}{\partial \delta} = - \frac{Q^0}{(1+\beta)Q^s} < 0, \text{ and}$$

$$\frac{\partial OER^*}{\partial \delta} = \frac{1}{\beta} > 0,$$

where $\mathbf{E} = \log(\alpha) + (1+\beta)\log(1+\beta) - \beta\log(\delta) - (1+\beta)\log(Q^0) + \beta\log(Q^S) - (\beta-\delta)$

Constraint Binding

1) Change of Q^0

$$\frac{\partial R^{**}}{\partial Q^0} = \mathbf{B} \quad \frac{\beta}{\beta-\delta} > 0,$$

$$\frac{\partial C^{**}}{\partial Q^0} = \mathbf{B} \quad \frac{\delta}{\beta-\delta} > 0,$$

$$\frac{\partial NOI^{**}}{\partial Q^0} = Q^S \left(\frac{\partial R^{**}}{\partial Q^0} - \frac{\partial C^{**}}{\partial Q^0}\right) = \mathbf{B}\, Q^S > 0,$$

$$\frac{\partial V^{**}}{\partial Q^0} = 0,$$

$$\frac{\partial IR^{**}}{\partial Q^0} = 0, \text{ and}$$

$$\frac{\partial OER^{**}}{\partial Q^0} = 0,$$

where $\mathbf{B} = \left(\frac{\delta^{\delta}(Q^0-Q^s)^{1-\beta+\delta}}{\alpha\beta^{\beta}}\right)^{1/(\beta-\delta)} > 0.$

2) Change of Q^S

$$\frac{\partial R^{**}}{\partial Q^s} = -\mathbf{B}\ \frac{\beta}{\beta-\delta} < 0,$$

$$\frac{\partial C^{**}}{\partial Q^s} = -\mathbf{B}\ \frac{\delta}{\beta-\delta} < 0,$$

$$\frac{\partial NOI^{**}}{\partial Q^s} = \mathbf{B}\ \left(Q^0 - Q^s\,\frac{1+\beta-\delta}{\beta-\delta}\right)(\beta^{\beta-\delta}-\delta^{\beta-\delta})^{1/(\beta-\delta)},$$

$$\frac{\partial V^{**}}{\partial Q^s} = 0,$$

$$\frac{\partial IR^{**}}{\partial Q^s} = 0, \text{ and}$$

$$\frac{\partial OER^{**}}{\partial Q^s} = 0.$$

3) Change of α

$$\frac{\partial R^{**}}{\partial \alpha} = -\mathbf{B}\ \frac{\beta(Q^0-Q^s)}{\alpha(\beta-\delta)} < 0,$$

$$\frac{\partial C^{**}}{\partial \alpha} = -\mathbf{B}\ \frac{\delta(Q^0-Q^s)}{\alpha(\beta-\delta)} < 0,$$

$$\frac{\partial NOI^{**}}{\partial \alpha} = Q^S\left(\frac{\partial R^{**}}{\partial \alpha} - \frac{\partial C^{**}}{\partial \alpha}\right) = -\mathbf{B}\ Q^S\ \frac{Q^0-Q^s}{\alpha} < 0,$$

$$\frac{\partial V^{**}}{\partial \alpha} = 0,$$

$$\frac{\partial IR^{**}}{\partial \alpha} = 0, \text{ and}$$

$$\frac{\partial OER^{**}}{\partial \alpha} = 0.$$

4) Change of β

$$\frac{\partial R^{**}}{\partial \beta} = \mathbf{B}\ (Q^0 - Q^s)\left(\frac{\beta}{(\beta-\delta)^2}\mathbf{F}+1\right),$$

$$\frac{\partial C^{**}}{\partial \beta} = \mathbf{B}\ \frac{\delta(Q^0-Q^s)}{(\beta-\delta)^2}\mathbf{F},$$

$$\frac{\partial NOI^{**}}{\partial \beta} = Q^s\left(\frac{\partial R^{**}}{\partial \beta}-\frac{\partial C^{**}}{\partial \beta}\right) = \mathbf{B}\ Q^s\ (Q^0 - Q^s)\left(\frac{\mathbf{F}}{\beta-\delta}+1\right),$$

$$\frac{\partial V^{**}}{\partial \beta} = 0,$$

$$\frac{\partial IR^{**}}{\partial \beta} = \frac{\delta}{\beta^2} > 0, \text{ and}$$

$$\frac{\partial OER^{**}}{\partial \beta} = -\frac{\delta}{\beta^2} < 0,$$

where $\mathbf{F} = \log(\alpha) - \log(Q^0 - Q^s) + \delta \log(\beta) - \delta \log(\delta) - (\beta - \delta)$.

5) Change of δ

$$\frac{\partial R^{**}}{\partial \delta} = -\mathbf{B}\ \frac{\beta(Q^0-Q^s)}{(\beta-\delta)^2}\mathbf{G},$$

$$\frac{\partial C^{**}}{\partial \delta} = -\mathbf{B}\ (Q^0 - Q^s)\left(\frac{\delta}{(\beta-\delta)^2}\mathbf{G}-1\right),$$

$$\frac{\partial NOI^{**}}{\partial \delta} = Q^s\left(\frac{\partial R^{**}}{\partial \delta}-\frac{\partial C^{**}}{\partial \delta}\right) = -\mathbf{B}\ Q^s\ (Q^0 - Q^s)\left(\frac{\mathbf{G}}{\beta-\delta}+1\right),$$

$$\frac{\partial V^{**}}{\partial \delta} = 0,$$

$$\frac{\partial IR^{**}}{\partial \delta} = -\frac{1}{\beta} < 0, \text{ and}$$

$$\frac{\partial OER^{**}}{\partial \delta} = \frac{1}{\beta} > 0,$$

where $\mathbf{G} = \log(\alpha) - \log(Q^0 - Q^s) + \beta \log(\beta) - \beta \log(\delta) - (\beta - \delta)$.