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"Planar Thinnest Deployment Pattern of Congruent Discs which Achieves 2-coverage"
by Ge Jun,
in his thesis presented at the School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China, June 2010.
Connected Optimal Two-Coverage of Sensor Networks

Ziqiu Yun, Jin Teng, Student Member, IEEE, Zuoming Yu, Dong Xuan, Member, IEEE, ACM, Biao Chen, Member, IEEE, ACM, and Wei Zhao, Fellow, IEEE

Abstract—In wireless sensor networks, multiple-coverage, in which each point is covered by more than one sensor, is often required to improve detection quality and achieve high fault tolerance. However, finding optimal patterns that achieve multiple-coverage in a plane remains a long-lasting open problem. In this paper, we first derive the optimal deployment density bound for two-covered deployment patterns where Voronoi polygons generated by sensor nodes are congruent. We then propose optimal two-coverage patterns based on the optimal bound. We further extend these patterns by considering the connectivity requirement and design a set of optimal patterns that achieve two-coverage and one-, two-, and three-connectivity. We also study optimal patterns under practical considerations. To our knowledge, our work is the very first that proves the optimality of multiple-covered deployment patterns.

Index Terms—Connected two-coverage.

I. INTRODUCTION

A. Motivation

Wireless sensor network (WSN) deployments with higher degrees of sensing coverage are important. Sensing coverage is closely coupled with monitoring quality. Some applications require multiple-coverage to achieve their goals, e.g., distributed target detection, data fusion, and classification [18], [25]. Moreover, multiple-coverage yields higher fault tolerance in the face of node failures.

Designing proper topology control and scheduling algorithms to achieve multiple-coverage in WSNs has attracted intensive research. However, one important cornerstone is still missing: What are the deployment patterns that use the minimum number of sensor nodes to cover every point of a region with at least k sensors (k-covered) for k > 1? This problem is also known as the optimal multiple-coverage problem.

Fundamentally, knowledge of optimal multiple deployment patterns will help reduce the number of required sensor nodes and their cost by avoiding ad hoc and potentially inefficient deployments. It can also provide theoretical guidance and performance measures for heuristic algorithms that are developed for topology control. Furthermore, such knowledge can help in designing practical guidelines for minimizing message collisions, improving network management, routing, etc. They will also provide guidelines in nonideal and more practical sensor deployment scenarios involving nonuniform communication/sensing ranges, nonuniformity with respect to hierarchical networks, certain gateway nodes, deployment fields, etc. It is noteworthy that studying optimal multiple-coverage patterns can help establish theoretical foundations and practical guidelines for not only WSNs, but also other wireless networks, such as mesh and cellular networks. For instance, the resulting optimal patterns can be directly applied to the deployment of access points (APs) in mesh networks and base stations (BSs) in cellular networks. In addition, the research results will broaden our understanding of the applications of geometry with respect to computer networks. In fact, the optimal one-coverage problem has already been intensively studied in recent years [1], [13], [26], [32].

The optimal multiple-coverage problem is related to the disk covering problem in the field of geometry. Notably, Kershner solved the optimal one-coverage problem in 1939 [16]. Kershner proved that the triangle lattice pattern is the optimal pattern that achieves one-coverage. The deployment density of the triangle lattice pattern is $2\pi/3\sqrt{3}$. However, the optimal multiple-coverage problem remains an open problem to date. In this paper, we study the optimal two-coverage problem as a first step to tackle this open problem.

B. Contributions

We highlight our two major contributions as follows.

Optimal Patterns to Achieve Two-Coverage: In this paper, for the first time, we provide the deployment density with proved optimality for two-covered deployment patterns. Specifically, we consider the deployment density of congruent deployments where Voronoi polygons generated by sensor nodes are congruent. We prove the following: The deployment density for all possible congruent deployments that achieve two-coverage is at least $4\pi/3\sqrt{3}$.

To the best of our knowledge, our results are the first direct answers to the optimal multiple-coverage problem. We are

1The formal definition of deployment density is provided in Section III.
now able to construct two-covered deployment patterns with proved optimality. In the triangle lattice pattern that is optimal to achieve one-coverage, the percentage of the overlapping areas is 17.3%. These overlapping areas in the triangle lattice pattern are already covered by at least two disks. Hence, one might naturally believe that such overlapping areas should be reused in constructing two-covered deployment patterns. However, notice that the optimal density we obtained is exactly twice the density of the triangle lattice pattern. This implies, surprisingly, that optimal two-covered deployment patterns do not reuse these overlapping areas in the triangle lattice pattern. If a two-covered deployment pattern is constructed by integrating two layers of triangle lattice patterns, the number of disks for covering a certain area is the sum of disk numbers in these triangle lattice patterns for covering this area. It follows that the density of this two-covered deployment is two times as large as the density of the triangle lattice pattern. Therefore, our results show that all deployment patterns constructed by integrating two layers of triangle lattice patterns, no matter how they overlap, have the proved optimal density. Fig. 1(a) illustrates one such example.

A successful separation from one two-covered deployment pattern into two one-covered deployment patterns can avoid reusing overlapping areas, which is a complicated problem, and hence provide much convenience. Although such separation has been assumed and used in existing works, it is solely based on researchers’ intuition. For the first time, our results in this paper provide solid theoretical support for this intuition. For example, in the ROAL protocol [17], sensor nodes are first selected to form a one-coverage pattern, and this process is repeated twice to eventually achieve two-coverage. Our results provide not only a theoretical basis but also performance measures for such separation-based algorithm designs. In general, our results are valuable to algorithm designs that are based on layer-wise separation to achieve multiple-coverage, such as [5], [10], etc.

Optimal Patterns to Achieve Connected Two-Coverage: In WSNs, connectivity is also an important requirement. Due to our results in this paper, it now becomes possible to obtain different two-covered deployment patterns with various degrees of connectivity by controlling the relative locations of two separated layers of one-coverage patterns.

We use the disk model for sensing in this paper. Each sensor is capable of detecting points only within distance \( r_s \), and we call \( r_s \) the sensing range of a sensor. Also, each sensor node is capable of communicating only with others within distance \( r_c \), and we call \( r_c \) the communication range of a sensor. In this paper, we give results for optimal deployment patterns that can achieve two-coverage and one-, two-, or three-connectivity. These results quite surprised us because situations are much more complicated than those of achieving one-coverage. Optimal congruent deployment patterns that can achieve one-coverage and one-, two-, or three-connectivity are always unique [1], [13], [26], [27], but \( c \)-optimal deployment patterns that can achieve two-coverage and one-, two-, or three-connectivity are only unique for certain values of \( \frac{r_s}{r_c} \), and for other values of \( \frac{r_s}{r_c} \), they contains infinitely many patterns.

Due to the infinite number of optimal connected two-covered deployment patterns, we can properly choose patterns that meet various practical requirements. For example, the deployment pattern in Fig. 1(c) has a larger distance between sensor nodes than the pattern in Fig. 1(b), which indicates potentially less communication interference.

Paper Organization: We present related work and some background information in Section II, followed by system models and definitions in Section III. In Section IV, we introduce our new results on optimal two-covered deployment patterns. In Section V, we consider connected two-coverage. Practical considerations are discussed in Section VI. We conclude the paper in Section VII.

II. RELATED WORK

The optimal deployment pattern problem is fundamental. Related work is scattered throughout both the areas of geometry and WSNs.

In geometry, this problem is related to the covering problem. Covering a set of points using a minimum number of given
geometric bodies has been extensively studied for disks in a large area [11], [16], disks on a bounded square [20], [21], orthogonal rectangles [9], fat convex bodies [7], [24], etc. Among these works, the most well-known and widely used result is Ker‐ shner’s, which states that the triangle lattice pattern is the optimal pattern to achieve one-coverage [16]. However, to the best of our knowledge, one-coverage dominates the geometry literature.

In the area of sensor networks, optimal deployment patterns to achieve one-coverage and connected one-coverage have been intensively studied. The triangle lattice pattern is optimal to achieve one-coverage and up to six-connectivity when \( r_c/r_s \geq \sqrt{3} \). This result is reproved in [32]. In 2005, the triangle pattern was shown to be nonoptimal [26] when \( r_c/r_s \geq \sqrt{3} \) is not satisfied; the strip-based pattern can outperform it when \( r_c = r_s \) [13]. In 2006, the asymptotic optimality of strip-based patterns to achieve full coverage and one- and two-connectivity was proved in [1] for all ranges of \( r_c/r_s \). In 2008, the asymptotically optimal pattern to achieve full coverage and four-connectivity for \( r_c/r_s > \sqrt{2} \) was proposed and proved to be optimal [3]. In [2], a complete set of deployment patterns for all ranges of \( r_c/r_s \) to achieve full-coverage and \( k \)-connectivity \((k \leq 6)\) was proposed. In this set of patterns, those for three- and five-connectivity were proved to be optimal among regular patterns for \( r_c/r_s \geq 1 \). Optimality is only conjectured for patterns to achieve three- and five-connectivity when \( r_c/r_s < 1 \), to achieve four-connectivity when \( r_c/r_s \leq \sqrt{2} \), and to achieve six-connectivity when \( r_c/r_s \leq \sqrt{3} \). The pattern to achieve six-connectivity when \( r_c/r_s \leq \sqrt{3} \) was provided, and its optimality among regular deployments was proved, in [27]. Recently, a pattern mutation phenomenon was discovered among optimal regular deployment patterns when \( r_c/r_s \) is small [4].

Considering multiple sensor node coverage, most work focuses on the issue of how to select the minimum number of sensors to be activated from a set of randomly predeployed sensors such that all interested discrete locations (or targets) are \( k \)-covered. This problem is known to be NP-hard [28]. Centralized and distributed approximation algorithms were then proposed [12], [23], [28], [31]. However, no existing work provides any leads toward optimal \( k \)-covered deployment patterns for \( k > 1 \).

III. MODEL AND PRELIMINARIES

There are several practical sensing models that stem from real device experiments. Megerian et al. [19] propose that the sensing quality can be expressed as \( \lambda/d^\alpha \), where \( \lambda \) and \( \alpha \) are sensor-dependent parameters and \( d \) is the distance between the sensor node and the detection target. In this model, the quality of sensing gradually attenuates with increasing distance. In [29], Zhou et al. propose a probabilistic sensing model. In this model, two values \( R_1 \) and \( R_2 \) \((R_1 \leq R_2)\) are defined from empirical observations. When the distance from the target to the sensor node is less than \( R_1 \), it will be detected with probability 1; when the distance is larger than \( R_2 \), the detection probability is 0; when the distance is between \( R_1 \) and \( R_2 \), the detection probability will exponentially decrease with increasing distance similar to that in [19].

We use the disk model for sensing in this paper. The disk model can be obtained from the above models by first setting a desirable threshold for sensing quality or probability, and then exploiting this threshold to determine the largest possible distance between the sensor and the target. This distance is then chosen to be the sensing range \( r_s \). We notice that for some types of sensors, the sensing capability may vary in different directions. We will discuss this in Section VI. The disk model provides a good abstraction from the real world that has been widely adopted, e.g., in [1] and [13]–[15], particularly when certain theoretical foundations are to be established.

We study the asymptotic optimality of deployment patterns, that is, a relatively large area compared to sensing and communication ranges is considered. The boundary effect is not important here and can be ignored. When the region boundaries have to be considered, optimality will vary from case to case as boundary shapes change. We study homogeneous wireless sensor networks where all sensor nodes are identical in terms of sensing.

**Definition 3.1 (Voronoi Polygon and Generating Point):** Let \( \{a_1, a_2, \ldots, a_p\} \) be a set of \( p \) points on a Euclidean plane \( S \). The Voronoi polygon \( V(a_i) \) is the set of all points in \( S \), which are closer to \( a_i \) (in terms of Euclidean distance) than to any other point, i.e., \( V(a_i) := \{x \in S : \forall j \in [1,p], d(x, a_i) \leq d(x, a_j)\} \), and point \( a_k \) is the generating point of \( V(a_i) \).

If two Voronoi polygons share an edge, their generating points are called neighboring generating points. In this paper, the generating point is always the very point locating a sensor.

Each Voronoi polygon edge resides on the perpendicular bisector of the line segment joining two neighboring generating points. Given any three generating points, the three perpendicular bisectors of the three line segments that join three pairs of the points, respectively, always intersect at one point. This point is a Voronoi polygon vertex. If there are \( n \) edges that intersect at a Voronoi polygon vertex, we say that the vertex is shared by \( n \) edges. Hence, any vertex in a Voronoi tessellation is shared by at least three edges.

From [1, Lemma 4.1], the average edge number of a Voronoi polygon in a Voronoi tessellation is not larger than six. Therefore, Voronoi polygons generated in a congruent deployment will have \( k \) edges for \( k = 3, 4, 5, \) or 6.

**Definition 3.2 (k-covered Congruent Deployment):** A \( k \)-covered congruent deployment is a sensor deployment pattern where the Voronoi polygons generated by sensors are all congruent and every point of the target region is covered by at least \( k \) sensors for \( k = 1, 2, 3, \ldots \). We call a \( k \)-covered congruent deployment a congruent deployment for short when the value of \( k \) is not important. We call a Voronoi tessellation a congruent Voronoi tessellation if the Voronoi polygons in the tessellation are all congruent. Congruent deployments generate congruent Voronoi tessellations with sensor nodes as generating points.

**Definition 3.3 (Deployment Density):** Deployment density, or density for short, is the ratio of the area of sensing disks to the area of Voronoi polygons generated by sensor nodes.

Since the area of the Voronoi polygons can be considered as individual contribution to coverage from each sensor node [1], smaller density implies a larger Voronoi polygon areas and means a more efficient deployment pattern.

**Definition 3.4 (\( e \)-Optimal Pattern):** A deployment pattern is called \( e \)-optimal if its deployment density equals the minimum density among all two-covered congruent deployments.
From Definition 3.4, a $c$-optimal pattern is not necessarily a congruent deployment pattern. It is a more general concept. A deployment pattern is called $c$-optimal as long as its density is equal to the density of the optimal two-covered congruent deployment patterns. On the other hand, the optimal two-covered congruent deployment patterns must be $c$-optimal. Moreover, when $\lim_{S \to 0} \frac{r^2}{S} = 0$, where $S$ is the measure of the area covered by sensors, we say a pattern is $c$-optimal if the limit of its deployment density is not greater than the minimum density among all two-covered congruent deployments.

Studying the optimal density of congruent deployments is meaningful both theoretically and practically. All proved globally optimal one-coverage patterns, which are optimal among all possible one-coverage patterns, are indeed congruent deployment patterns. Examples include the triangle lattice pattern [16], the strip-based pattern [1], and the diamond pattern [3]. Congruent deployment patterns also have strong practical implications in homogeneous WSNs. Studying $c$-optimal patterns is an important step toward globally optimal two-coverage solutions.

IV. TWO-COVERAGE

We present our results on two-coverage in this section. Theorem 4.1 is the main conclusion.

Theorem 4.1: $c$-optimal deployment patterns have density $d^* = 4\pi / 3\sqrt{3}$.

Theorem 4.1 provides a direct answer to the optimal multiple-coverage problem for the first time by stating that the smallest density for any two-covered congruent deployment is $d^*$. It also provides a valuable criterion for optimality determination.

The proof of Theorem 4.1 consists of two steps. In the first step, we find all possible congruent Voronoi tessellations that are generated by congruent deployment patterns. In the second step, we prove that the densities of these congruent deployment patterns are at least $d^*$ if two-coverage is achieved. Sections IV-A and IV-B detail these steps.

A. Case Reduction

The following Lemma 4.1 summarizes the results of the first step. It states that we can classify all possible congruent Voronoi tessellations into 12 types.

Lemma 4.1: There are 12 types of congruent Voronoi tessellations.

The 12 types of congruent tessellations, types (a)–(l), are shown in Fig. 2(a)–(l). Here, we provide the descriptions for 12 types of congruent Voronoi tessellations as shown in Fig. 2(a)–(l).

Type (a): Voronoi polygons generated by sensors are triangles.

Type (b): As shown in Fig. 3(a), three Voronoi polygons in this type can construct a regular triangle. Each Voronoi polygon has one interior angle of $\pi / 3$ and one of $\pi / 6$. The three generating points (sensor nodes) in such three Voronoi polygons are the vertices of a regular triangle.

Type (c): Each Voronoi polygon is a right angle trapezoid. Four Voronoi polygons in this type can construct a rectangle.

Type (d): In each Voronoi polygon, there are two opposite interior angles that are right angles. The two edges of one of these two right angles are of the same length. Four such Voronoi polygons construct a square.

Type (e): The Voronoi polygons are isosceles trapezoids.
Type (h): As shown in Fig. 4(a), in each Voronoi polygon, e.g., $ABCD$, of this type, there are two opposite interior angles that are right angles. The two edges of each non-right interior angle are of the same length. It generating point lies on the longer diagonal.

Type (i): As shown in Fig. 4(b), in each Voronoi polygon $ABCD$, $\angle A = \angle B = \angle C = \angle D = 2\pi/3$ and $OA = OD = 2AB = 2BC = 2CD$. Generating point $P$ is the middle point of $AD$.

Type (j): Voronoi polygons are pentagons each with two neighboring interior angles being right angles.

Type (k): Each Voronoi polygon in this type consists of one square and one isosceles right-triangle.

Type (l): As shown in Fig. 4(c), Voronoi polygon $ABCD$ is an inscribed hexagon (of some circle) with the generating point being the center of the circle. In Voronoi polygon $ABCD$, $AB = DB$, $AH = DE$, $BC = AF$, $HC = AF$, $CD = DF$, and $CD = DF$.

Types (a)-(l) can be divided into two groups. The first group contains types (c), (d), (i), and (k). In this group, each type only contains one specific pattern form, i.e., the interior angles of polygons in each type cannot change once $r_4$ is fixed. The second group contains the remaining types. In this group, each type contains infinitely many pattern forms, i.e., interior angles and edge lengths of polygons in each type can change even if $r_4$ is fixed. For example, both the triangle lattice pattern and the strip-based pattern [1] are special forms of type (l), and both the square pattern (the pattern in which Voronoi polygons are squares) and the rectangle pattern (the pattern in which Voronoi polygons are rectangles) can be special forms of types (e), (f), (g), and (h). Although Voronoi polygons in type (k) patterns apparently are special forms of those in type (j) patterns, these two types are in fact different. In type (k), each Voronoi polygon is a combination of a square and a right-angled isosceles triangle, whereas in type (j), we can change the interior angles except two right angles. The relationship between types (c) and (h) is similar.

Here, we provide the proof for Lemma 4.1. We start our proofs with the following lemmas.

**Lemma 4.2:** If a vertex $O$ in a Voronoi tessellation is shared by four edges, the opposite angles with $O$ as their vertex are complementary.

**Proof:** In Fig. 5(a), we have $\angle P_1OB = \angle P_2OB$, $\angle P_3OC = \angle P_4OC$, $\angle P_5OD = \angle P_6OD$, and $\angle P_7OA = \angle P_8OA$. Hence, $\angle AOB + \angle COD = \angle BOC + \angle DOA$. Since $\angle AOB + \angle COD + \angle BOC + \angle DOA = 2\pi$, $\angle AOB + \angle COD = \angle BOC + \angle DOA = \pi$.

**Lemma 4.3:** In a Voronoi tessellation, the structure shown in Fig. 6(b) cannot exist unless two line segments are perpendicular to each other.

**Lemma 4.4:** If a congruent Voronoi tessellation consists of quadrilaterals, no vertex is shared by five edges or $k$ edges for $k > 6$.

**Proof:** Denote the four angles in order of a Voronoi quadrilateral by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\alpha_4$ corresponding to vertices $u, b, c$, and $d$. From [1, Lemma 4.1], the average number of edges shared by one vertex is four. If there are vertices that are shared by five edges, then there should be the same number of vertices that are shared by three edges. There are three angles at a vertex when this vertex is shared by three edges. The sum of them is $2\pi$. Since all of them are taken from angles $\alpha_1, \alpha_2, \alpha_3,$ and $\alpha_4$, two of the three angles must be the same since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. Without loss of generality, we denote the three angles are $\alpha_1, \alpha_1,$ and $\alpha_2$. In the following, we first show there cannot be any vertex that is shared by four edges when there are vertices that are shared by five edges and three edges. From $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$ and $2\alpha_1 + \alpha_2 = 2\pi$, we have $\alpha_1 = \alpha_3 + \alpha_4$. Hence, $2\alpha_1 \neq \pi, \alpha_1 + \alpha_2 \neq \pi,$ and $\alpha_2 + \alpha_3 \neq \pi$. If there are vertices that are shared by four edges, from Lemma 4.2, there are three cases for the two pairs of complementary opposite angles at such vertices:

1) $\alpha_1, \alpha_3$ and $\alpha_2, \alpha_4,$ or $\alpha_1, \alpha_4$ and $\alpha_2, \alpha_3$;
2) $\alpha_2, \alpha_2$ and $\alpha_1, \alpha_3,$ or $\alpha_2, \alpha_2$ and $\alpha_1, \alpha_4$;
3) $\alpha_2, \alpha_3$ and $\alpha_2, \alpha_4,$ or $\alpha_2, \alpha_2$ and $\alpha_4, \alpha_4$.

From case 1), it is easy to have $\alpha_3 = \pi/2$ and $\alpha_4 = 0$, which is impossible. Similarly, cases 2) and 3) are also impossible. When the three angles at the vertex shared by three edges are denoted by $\alpha_1, \alpha_1$, and $\alpha_2$, then the five angles at the vertex shared by five edges are $\alpha_2, \alpha_3, \alpha_3, \alpha_4$, and $\alpha_5$. We have two cases for the order of these five angles:

1) $\alpha_2, \alpha_3, \alpha_3, \alpha_4$ or $\alpha_2, \alpha_2, \alpha_2, \alpha_4, \alpha_3$;
2) $\alpha_2, \alpha_3, \alpha_3, \alpha_4, \alpha_5$ or $\alpha_2, \alpha_2, \alpha_2, \alpha_4, \alpha_3$.

The proofs for impossibility of these two cases are the same. Consider case 1). Consider the Voronoi quadrilateral that contains $\alpha_2$. The neighboring edges of $\alpha_2$ in this quadrilateral is $ab$ and $bc$. Since $ab$ is actually edge $c_0$ or $c_b$ of the Voronoi quadrilateral on the other side of $ab$, the length of $ab$ is equal to $c_a$ or $c_b$. Similar analysis applies to $bc$. Then, all Voronoi quadrilaterals are rhombi and then $\alpha_1 + \alpha_2 = \pi$. As $2\alpha_1 + \alpha_2 = 2\pi$, we have $\alpha_1 = \pi$ and $\alpha_2 = 0$, which is impossible. The proof of the case when there is no vertex that is shared by $k$ edges for $k > 6$ is similar.

**Lemma 4.5:** The tessellation shown in Fig. 6(a) is not a congruent Voronoi tessellation. [Any congruent Voronoi tessellation will not contain the structure shown in Fig. 6(b).]
Proof: Assume the tessellation shown in Fig. 6 is a congruent Voronoi tessellation. Take a Voronoi parallelogram between lines $l_1$ and $l_2$ and assume its generating point is $P_1$. Denote the dashed line that goes through $P_1$ and is parallel to $l_1$ by $l_1'$. $l_1$ intersects edges to the right side of $A_1, A_2, \ldots, A_k$. Denote the edges on which $A_k$ is by $e_k$. We can then denote a point that is the symmetry point of $P_1$ about edge $e_1$ by $P_2$, a point that is the symmetry point of $P_2$ about edge $e_2$ by $P_3$, $\ldots$, and a point that is the symmetry point of $P_k$ about edge $e_k$ by $P_{k+1}$. From the definition of Voronoi polygon, $P_k, k \geq 2$, are the generating points of Voronoi parallelogram with edges $e_{k-1}$ and $e_k$. Now we have $P_1P_{2n+1} = 2n\alpha \sin \alpha$. Denote the distance between $P_{2n}$ and $l_1$ as $d_k$. Then, $d_{2n+1} = P_1P_{2n+1} \cos \alpha + d_1 = d_1 + n\alpha \sin 2\alpha$. Notice $d_{2n+1} \to \infty$ as $n \to \infty$. Hence, there is some $n$ that places $P_{2n}$ outside of the area between $l_1$ and $l_2$. This contradicts to $P_k$ being a generating point for any $k > 2$.

Lemma 4.6: If a vertex $O$ in a Voronoi tessellation is shared by three edges $OA, OB,$ and $OC$, then $\angle P_1OA + \angle BOC = \pi$, where $P_1$ is the generating point for the Voronoi polygon that consists of edges $OA$ and $OB$.

Lemma 4.6 tells that the lines that go through generating points can be decided given the position of three edges that share a vertex. This lemma can be directly derived from the contents of [22, Section 2.3].

We make the following three propositions.

Proposition 4.1: If the Voronoi polygons generated in a congruent deployment are quadrilaterals, there are seven types of tessellations. They are shown in Fig. 2(b)–(h).

Proposition 4.2: If the Voronoi polygons generated in a congruent deployment are pentagons, there are three types of tessellations. They are shown in Fig. 2(i)–(k).

Proposition 4.3: If the Voronoi polygons generated in a congruent deployment are hexagons, there is one tessellation shown in Fig. 2(l).

Though we can offer direct proofs of the above propositions, we omit the proofs because of page limitation and the fact that there exists another way to prove them. From the contents in Grünbaum and Shephard’s book [8, pp. 475–481], there are 56 types of tessellations that consists of congruent quadrilaterals, 24 types for congruent pentagons, and 13 types for congruent hexagons. Then, we can obtain above propositions by applying our lemmas to eliminate those that cannot be Voronoi tessellations and combine several types into a general type.

Lemma 4.1 is the combination of Propositions 4.1–4.3.

B. Bound Derivation

In this section, we prove that there exists a lower bound of all deployment density for all the deployment patterns that can generate one of the 12 types of congruent Voronoi tessellations in Lemma 4.1 and simultaneously achieve two-coverage. Lemma 4.7 holds the key to the second step.

Lemma 4.7: The deployment density of any two-covered congruent deployment pattern that can generate one of 12 types of congruent Voronoi tessellations is at least $d^*$. Proof: Lemma 4.7 holds clearly for type (a) shown in Fig. 2(a), i.e., when the congruent Voronoi tessellation consists of triangles, since the area of a triangle within a disk reaches its maximum when it is regular and inscribed in that disk. To prove Lemma 4.7, we need to prove that the density is at least $d^*$ for deployments that generate the remaining 11 types.

To show that the density of each of the remaining 11 Voronoi tessellation types is at least $d^*$ for two-covered deployments, the proof techniques we use differ depending on types. For types (e), (f), (j), and (k), we need to first transform Voronoi polygons while keeping their areas unchanged, then construct a function of area, and get the minimum value of this function, and finally prove this value is at least $d^*$. For type (g), we need to divide the two-coverage pattern into two one-coverage patterns. For other types, we just use the geometrical characteristics of these patterns.

Consider type (l) shown in Fig. 2(b). As shown in Fig. 3(a), $ABC$ is a regular triangle. $O$ is the center point of the triangle. Let the edge length be 1. \( DC = EA = FB = a \). \( a < 1 \). Let $P$ be the generating point of polygon $DCEO$. Since polygon $DCEO$ is within the disk that centers at $P$, radius $r$ should satisfy

$$r \geq \max\{PD, PC\} \geq \frac{1}{2}(PD + PC) \geq \frac{1}{2}DC = \frac{a}{2}.$$  

Hence, density

$$d = \frac{\pi r^2}{\frac{a^2}{4} - \frac{a}{3}} \geq \frac{\pi (\frac{a}{2})^2}{\frac{a^2}{4} - \frac{a}{3}} \geq 4\sqrt{3}\pi \left(\frac{1}{2} \times \frac{2}{3}\right)^2 = \frac{4\pi}{3\sqrt{3}}.$$  

Consider type (c) shown in Fig. 2(c). Voronoi polygon $ABCD$, as shown in Fig. 3(b), satisfies $\angle BAC = \pi/3, \angle BCD = 2\pi/3, \angle ABC = \angle ADC = \pi/2$. Let $BC$ be 1. From Lemma 4.6, the generating point $P_1$ for $ABCD$ is on $AC$. Let $P_1 = x$, then $CP_1 = 2 - x$. Since $d = \pi r^2/\sqrt{3}$, if $d \leq d^* = 4\pi/3\sqrt{3}$, then $r \leq 2/\sqrt{3}$. To achieve one-coverage, radius $r$ should satisfy $r \geq \max\{x, 2 - x\}$. From $\max\{x, 2 - x\} \leq 2/\sqrt{3}$, we have $2 - 2/\sqrt{3} \leq x \leq 2/\sqrt{3}$.

Denote the symmetry point of $P_1$ about $AB$ be $P_2$, the symmetry point of $P_1$ about $BC$ is $P_3$. Then, $P_2$ and $P_3$ are also generating points according to the definition of Voronoi tessellation. That is, $P_2$ and $P_3$ are the center points of disks that intersect with the disk centering at $P_1$. To achieve two-coverage, $AB$ should be fully covered by the disks centering at $P_2$ and $P_3$. We then have

$$x + \sqrt{r^2 - \left(\frac{\sqrt{3}x}{2}\right)^2} + \sqrt{r^2 - \left(\frac{\sqrt{3}(2 - x)}{2}\right)^2} - \frac{2 - x}{2} \geq 2.$$  

Then, we obtain

$$r^2 \geq \left(\frac{x^2 - 9x + 12}{2(3 - x)}\right)^2 + \frac{3}{4}x^2.$$
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Fig. 7. Reference figures used in the proof of Theorem 4.1. (a) is for type \(\star\). (b) is for type \(\dagger\). (c) is for type \(\ddagger\).

The right side of the above inequation is a monotone increasing function of \(x\) in the range of \([2 - 2/\sqrt{3}, 2/\sqrt{3}]\). Hence
\[
r \geq \sqrt{\left(\frac{x^2 - 9x + 12}{2(3-x)}\right)^2 + \frac{3x^2}{4}} \approx 1.3118 > \frac{2}{\sqrt{3}}.
\]
Hence, \(d > d^*\).

Consider type (d) shown in Fig. 2(d). \(ABCD\) is a rhombus with \(\angle BAD = \pi/3\). Let \(AB = 1\). Let \(P\) be the center point of a disk that covers \(ABCD\) and \(P\) be within \(ABCD\). The radius \(r\) should satisfy
\[
r \geq \max\{PA, PC\} \geq \frac{1}{2}(PA + PC) > \frac{1}{2}AC = \frac{\sqrt{3}}{2}.
\]
Then, we have
\[
d = \frac{\pi r^2}{\sqrt{3}} > \frac{\pi (\sqrt{3}/2)^2}{\sqrt{3}} = \frac{\pi \sqrt{3} \pi}{2} > 4\pi \frac{3\sqrt{3}}{3} = d^*.
\]
Consider type (e) shown in Fig. 2(e). We can transform the Voronoi polygon that is a right trapezoid to a rectangle while keeping the area unchanged [Fig. 7(a)]. Thus, we can focus on the cases where the Voronoi polygons are rectangular.

Since the density will not change when the location of the generating point changes within the Voronoi rectangle, we may consider the case where the generating point is the central point of the rectangle. Denote the length of the short edges of the rectangle by \(a\). Note that, in a two-covered deployment, for each Voronoi polygon generated by a sensor node, there is at least one edge to which the distance from the sensor node is at most \(r/2\). Hence, we have \(a \leq r_1\). Due to the two-coverage requirement, the Voronoi rectangle with generating point \(P\) should be fully covered by other four disks with centers \(P_1, P_2, P_3,\) and \(P_4\), as shown in Fig. 3(a). We can obtain the length of the long edges as \(b \leq r_1 + \sqrt{r_1^2 - a^2}\). Hence, the area of the rectangle is at most \(a(b) = a(r_1 + \sqrt{r_1^2 - a^2})\). We can obtain that \(f(a)\) achieves its maximum in the range of \([0, r_1]\) when \(a = \sqrt{3}r_1/2\). The maximal value is \(3\sqrt{3}r_1/4\). Hence, the density is not less than \(d^*\).

Consider type (f) shown in Fig. 2(f). Since \(\angle ADC = \angle ACB = \pi/2\), vertices \(A, B, C,\) and \(D\) reside on the circle that centers at the middle point of \(AC\) denoted by \(P\). \(P\) is also the generating point of \(ABCD\). Let the radius of this circle be \(1\) and \(\angle CAB = a\). Then, the density is \(d = \pi r^2/(1 + \sin 2a)\). In the following, we show \(d \geq d^*\) by contradiction.

Assume \(d < d^*\). Since polygon \(ABCD\) is the Voronoi polygon of the circle with center \(P\) and radius \(r\), we have \(r \geq PA > 1\). Hence
\[
\frac{\pi}{1 + \sin 2a} = \frac{\pi r^2}{1 + \sin 2a} < d = \left(\frac{4\pi}{3\sqrt{3}}\right)^2.
\]
That is, \(\sin 2a > 3\sqrt{3}/4 - 1\). Denote \(3\sqrt{3}/4 - 1\) by \(S_1\). Then, \(\cos 2a = \frac{1 - \sin^2 2a}{\cos^2 2a} < \frac{1 - S_1^2}{S_1}\). Denote \(1 - S_1^2\) by \(T_1\). Because two-coverage should be achieved, \(AC\) must be covered by the two circles with radius \(r\) that center at \(P_1\) and \(P_2\). Then, we have
\[
\cos 2a + \sqrt{r^2 - \sin^2 2a} + \sqrt{r^2 - 1} \geq 2.
\]
Then
\[
T_1 + \sqrt{r^2 - S_1^2} + \sqrt{r^2 - 1} > 2.
\]
Let \(r_1\) be the positive root of \(T_1 + \sqrt{r^2 - S_1^2} + \sqrt{r^2 - 1} = 2\), then \(r > r_1 > 1\). Hence, we have
\[
\frac{\pi r_1^2}{1 + \sin 2a} < d = \left(\frac{4\pi}{3\sqrt{3}}\right)^2.
\]
Let \(T_2\) be the positive root of \(T_2 + \sqrt{r^2 - S_2^2} + \sqrt{r^2 - 1} = 2\), then \(T_2 < T_1\). Let \(r_2\) be the positive root of \(T_2 + \sqrt{r^2 - S_2^2} + \sqrt{r^2 - 1} = 2\), then \(r > r_2 > r_1 > 1\). We continue in this way. That is, for \(r_k = 1\) and \(k = 1, 2, 3, \ldots\), let \(S_k = 3\sqrt{3}/4 - 1/4 = 1/4\). Then, \(r_k = 1 - S_k^2\) and \(r_k\) is the positive root for equation \(T_k + \sqrt{r^2 - S_k^2} + \sqrt{r^2 - 1} = 2\), then \(r > r_k > r_k - 1 > \cdots > r_1 > r_0\). When \(k > 12\), \(r_k > 1.3\). Hence
\[
d = \frac{\pi r_1^2}{1 + \sin 2a} > \left(\frac{4\pi}{3\sqrt{3}}\right)^2 > d^*.
\]
which is contradictory to the assumption.

Consider type (g) shown in Fig. 2(g). As shown in Fig. 7(b), we divide generating points into two groups denoted by \(\star\)’s and \(\odot\)’s. We connect those denoted by small crosses, which partitions the plane into a tessellation of congruent triangles. One such triangle is denoted by \(ABC\) in Fig. 7(b). Consider triangle \(ABC\). We remove the disk centered at the generating point \(D\) within \(ABC\). If we remove any one sensor from a two-covered deployment, the remaining sensor nodes achieve at least one-coverage. Hence, the remaining disks still achieve one-coverage here. Since the minimum distance from any point within triangle \(ABC\) to its vertices is at most the distance from the three \(ABC\) vertices to any other generating point except \(D\), triangle \(ABC\) can be fully covered by three disks centered at the vertices. Note that, given a triangle, a necessary and sufficient condition for the disks with radius \(r_1\) centered at its vertices to...
cover it is that the radius of the circumscribed circle of the triangle is at most \( r_* \). Therefore, all the triangles whose vertices denoted by small crosses are fully covered by disks centered at their vertices. Hence, the entire plane is one-covered by the disks centered at the vertices denoted by small crosses. For exactly the same reason, the entire plane is also one-covered by the disks center at the vertices denoted by small circles. If a two-covered deployment is constructed with two one-coverage layers, its density is at least \( 4\pi/3\sqrt{3} \). Hence, the density here is at least \( d^* \).

Consider type \((h)\) shown in Fig. 2(h). From Lemma 4.3, there are two opposite angles in \( ABCD \) that equal \( \pi/2 \). As shown in Fig. 4(a), \( ABCD, AB = AD, BC = CD, \angle ABC = \angle ADC = \pi/2 \). Let \( AB = 1 \) and \( BC = a \).

From the definition of Voronoi polygon, the center of the disk is at the middle of the long diagonal. To achieve two-coverage, \( DC \) should be fully covered by the disks centering at \( P_1 \) and \( P_2 \). We then have

\[
2\sqrt{r^2 - \left( \frac{a}{\sqrt{a^2 + 1}} \right)^2} \geq \sqrt{a^2 + 1}
\]

then

\[
r^2 \geq \frac{a^2}{a^2 + 1} + \frac{a^2 + 1}{4}.
\]

We obtain

\[
d = \frac{\pi r^2}{a} \geq \pi \left( \frac{a}{a^2 + 1} + \frac{a^2 + 1}{4a} \right) > \pi > \frac{4\pi}{3\sqrt{3}} = d^*.
\]

Consider type \((i)\) shown in Fig. 2(i). From Lemma 4.6, in Fig. 4(b), \( \angle A = \angle B = \angle C = \angle D = 2\pi/3, OA = OD = 2AB = 2BC = 2CD \). The generating point, \( P \), is the middle point of \( AD \). To achieve two coverage, length of \( OP \) cannot be larger than \( r \). It follows that the maximal measure of pentagon \( OABC \) is \( \frac{12\sqrt{3}}{7} \). Therefore, we have

\[
d \geq \frac{4\sqrt{3}\pi}{7} > d^*.
\]

Consider types \((j)\) and \((k)\) shown in Fig. 2(j) and (k), respectively. The proof is the same as that for type \((e)\) (each Voronoi polygon here can be considered as two right angle trapezoids as those in type \((e)\)) by transforming the Voronoi polygons into rectangles with the same area.

Consider type \((l)\) shown in Fig. 2(l). As shown in Fig. 7(c), denote the generating point \( P \) within polygon \( ABCD \) by \( P \). Note that if a vertex \( O \) in a Voronoi tessellation is shared by three edges \( OA, OB, \) and \( OC \), then \( \angle P_OA + \angle BOC = \pi \), where \( P_1 \) is the generating point of the Voronoi polygon that consists of edges \( OA \) and \( OB \). Hence, we have \( AB \parallel DE, AB = DE, BC = AF, \) and \( CD = EF \). Fig. 7(c) is a Voronoi polygon iff \( ABCD \) inscribes a disk that is centered at \( P \).

Denote the central point \( P \) that is opposite to the longest edge of polygon \( ABCD \) by \( P \). We have \( \pi/3 \leq \alpha < \pi \). To achieve 2-coverage, the sensing disk radius \( r \) should satisfy \( r > 2 \cos \alpha/2 \). Hence, \( d \geq \pi r^2 / (\sin \alpha + 2 \sin \alpha/2) \geq 2\pi (\cos \alpha + 1)/(\sin \alpha + 2 \cos \alpha/2) \). Since \( f(\alpha) = \sin \alpha + 2 \cos \alpha/2 \) is a monotone decreasing function at range \( [\pi/3, \pi] \), we have \( d > 2\pi/3\sqrt{3} = d^* \).

Since the lower bound \( d^* \) obtained in Lemma 4.7 is achievable, e.g., by integrating two layers of triangle lattice patterns, \( d^* \) is the optimal density of two-covered congruent deployment patterns. Theorem 4.1 then holds directly. We also conjecture that this density \( d^* \) is actually a globally optimal density since all proved one-coverage optimal patterns are indeed congruent deployment patterns, as we mentioned before in Section III.

Based on Theorem 4.1, it now becomes possible to determine optimality for multiple-covered deployments. Theorem 4.2 provides theoretical support by stating the optimality for arbitrarily overlapping of two layers of triangle lattice patterns. Its proof is straightforward and omitted here.

**Theorem 4.2:** All deployments constructed with two layers of triangle lattice patterns are \( c \)-optimal.

We can naturally obtain another important conclusion in Theorem 4.3. Theorem 4.3 states that, among all two-covered congruent deployments, those constructed with two layers of triangle lattice patterns are optimal.

**Theorem 4.3:** All congruent deployment patterns constructed with two layers of triangle lattice patterns are optimal congruent deployment patterns.

There are infinitely many congruent deployments as long as triangle edges in different layers are parallel. Hence, we have infinitely many optimal two-covered congruent deployments. This contrasts with the one-coverage case where it was strictly proved by Kershner in [16] that the triangle lattice pattern is the unique optimal congruent deployment.

V. CONNECTED TWO-COVERAGE

In WSNs, connectivity is also important. The disk model is widely used in the literature [1], [13], [26], [32]. It can be considered a simplified version of the real models, e.g., the one proposed by Zuniga et al. [30]. They suggest communication link quality can be measured by the packet reception rate (PRR). The PRR at distance \( d \) is \( (1 - e^{-\frac{d}{P_r(B_L - \sqrt{2}d^2)}}/2)^{0.5} \), where \( P_r \) is the output power of the transmitter, \( P_L(d) \) is the path loss at distance \( d \), \( P_n \) is the noise floor, and \( L \) is the frame length. The disk model considers a connection established between two sensor nodes only if the PRR from each other is above a certain threshold.

Our results in Section IV provide a basis to design connected \( c \)-optimal patterns. When \( \sqrt{\frac{2}{\pi} - 1} \) is big enough, to obtain different degrees of connectivity, our approach consists of two steps. The first step is to design the relative position of two layers of triangle lattice patterns according to the relationship between \( r_* \) and \( r_* \). The second step is to integrate the two layers together.

From Theorem 4.2, the deployment patterns constructed in this way are all \( c \)-optimal. When \( \sqrt{\frac{2}{\pi} - 1} \) is not big enough, we choose a pattern from 12 types of congruent Voronoi tessellations listed in Lemma 4.1 and prove this pattern is optimal that achieves the degree of connectivity which we need. Same as in [1] and [2], when considering one- or two-connectivity, we assume that \( \lim_{\frac{d}{r}} = 0 \), where \( S \) is the measure of the area covered by sensors.

We skip the obvious cases, e.g., 12-connectivity can be achieved when vertices of two triangle lattice layers entirely overlap and \( r_* > \sqrt{3} \). In the following, we present general forms of two-covered deployment patterns that achieve one-, two-, and three-connectivity, respectively.
A. Optimal One- or Two-Connected Two-Coverage Patterns

In this section, we present optimal one- or two-connected c-optimal deployment patterns for $r_c < \sqrt{3}r_s$.

**Theorem 5.1:** When $\sqrt{3}r_s/2 < r_c < \sqrt{3}r_s$, there are infinitely many c-optimal deployment patterns that can achieve two-coverage and one- or two-connectivity; when $r_c \leq \sqrt{3}r_s/2$, the rectangle pattern (i.e., pattern the Voronoi polygons of which are congruent rectangles) is the only c-optimal deployment patterns that can achieve two-coverage and one- or two-connectivity.

**Proof:** When $\sqrt{3}r_s/2 < r_c < \sqrt{3}r_s$, the general form of optimal patterns can be illustrated by showing how they are constructed. To obtain such patterns, we integrate two triangle lattice layers as shown in Fig. 8(a). In each layer, we first let each triangle have a horizontal edge, e.g., edge $BC$, and then make a small disk with radius $r_c - \sqrt{3}r_s/2$ that is centered at the middle point of the horizontal edge of the triangle. We then let each triangle in the other layer also have a horizontal edge and each vertex (denoted by •’s) be within each small disk in the first layer.

In the final deployment pattern we construct, the distance between any vertex in the second layer, e.g., node $A$, and the corresponding vertices in the first layer, e.g., nodes $B$ and $C$, are both less than $r_c$ because the sum of two edges in a triangle is larger than the other. Hence, such deployment can achieve two-coverage and one-connectivity when one vertical line of sensor nodes is deployed to connect all horizontal sensor node lines, and two-connectivity if two vertical lines of sensor nodes are deployed to connect all horizontal lines at the boundaries.

When $r_c - \sqrt{3}r_s/2$, to obtain patterns that achieve two-coverage and one- or two-connectivity, we can only put each vertex of triangles of the second layer at the middle point of the horizontal edge in the first layer. A typical two-coverage and one-connectivity deployment pattern is shown in Fig. 1(b), and a typical two-connectivity pattern is shown in Fig. 8(b). Both of them are special forms of Types ($e$), ($f$), ($g$), and ($h$). In these two specific deployment patterns, Voronoi polygons are congruent rectangles except at the left boundary in one-connectivity case, and except at the left and right boundaries in two-connectivity case. In these patterns, the ratio of lengths of two sides of rectangles is $1 : \sqrt{3}$.

On the other hand, a necessary condition of achieving one- or two-connectivity is that there are two neighbors of every sensor, such that the distance between this sensor and each neighbor is not greater than $r_c$. Checking patterns with density $d^*$ among the 12 types of congruent Voronoi tessellations listed in Fig. 2 other than the above-mentioned rectangle pattern, no pattern satisfies this condition. Therefore, the above-mentioned rectangle pattern is the only c-optimal deployment pattern that can achieve two-coverage and one- or two-connectivity when $r_c = \sqrt{3}r_s/2$.

When $r_c < \sqrt{3}r_s/2$, to achieve two-coverage and one- or two-connectivity, we can change equilateral triangles into isosceles triangles with a height of $r_c + \sqrt{r_c^2 - r_s^2}$ and a bottom of $2r_s$, in the first layer, and put each vertex of triangles of the second layer at the middle point of the horizontal edge in the first layer. The Voronoi polygons of this pattern are congruent rectangles and the length of the two sides of them are $r_c$ and $r_s + \sqrt{r_c^2 - r_s^2}$, and hence the measure of a Voronoi polygon in this pattern is $r_s(r_s + \sqrt{r_c^2 - r_s^2})$. Notice the above-mentioned necessary condition of achieving one- or two-connectivity, and with a routine calculating, we can show that to satisfy this condition, the measure of a Voronoi polygon of each pattern among the 12 types of congruent Voronoi tessellations other than the rectangle pattern is smaller than $r_c(r_s + \sqrt{r_c^2 - r_s^2})$ except that of Type ($e$). In Type ($e$), as shown in Fig. 9(a), when we move sensors close enough to the points shared by six Voronoi polygons, each sensor can has two neighbors such that the distance between this sensor and each neighbor is not greater than $r_c$ while keeping the Voronoi polygons unchanged. However, six such sensors—one each is a neighbor of two others—make a circle. To achieve one- or two-connectivity, the distance between the two closest circles cannot be greater than $r_c$, and the latter condition makes the measure of a Voronoi polygon in this pattern smaller than $r_c(r_s + \sqrt{r_c^2 - r_s^2})$. Thus, the rectangle pattern is the only c-optimal deployment patterns that can achieve two-coverage and one- or two-connectivity when $r_c < \sqrt{3}r_s/2$.

We have one remark to make about Theorem 5.1. The optimal pattern for $r_c < \sqrt{3}r_s/2$ is different from our intuition. The striped-based patterns [1], [13], which are special forms of Type ($l$) and are optimal for one- or two-connectivity and one-coverage, are not c-optimal deployment patterns that...
can achieve two-coverage and one- or two-connectivity. As shown in Fig. 9(b), in the striped-based patterns, the sensors form isosceles triangles. To achieve two-coverage and one- or two-connectivity, the height of these triangles cannot be greater than $\sqrt{r_s^2 - r_c^2} + \sqrt{r_s^2 - r_c^2}/4$, and the length of these triangles cannot be greater than $r_c$. It follows that the measure of a Voronoi polygon in this pattern is $r_c\left(\sqrt{r_s^2 - r_c^2}/4 + \sqrt{r_s^2 - r_c^2}\right)$, which is smaller than that in rectangle pattern. Therefore, striped-based patterns are not $c$-optimal deployment patterns that can achieve two-coverage and one- or two-connectivity.

B. Optimal Three-Connected Two-Coverage Patterns

In this section, we present two-covered and three-connected $c$-optimal patterns for $r_c < \sqrt{3} r_s$.

**Theorem 5.2:** When $r_s < r_c < \sqrt{3} r_s$, there are infinitely many $c$-optimal deployment patterns that can achieve two-coverage and three-connectivity; when $r_c \leq r_s$, the equilateral triangle pattern (i.e., pattern of the Voronoi polygons that are equilateral triangles) is the only $c$-optimal deployment pattern that can achieve two-coverage and three-connectivity.

**Proof:** Again, when $r_c < r_s < \sqrt{3} r_s$, we will explain how to construct the general form of such optimal patterns. Two triangle lattice layers are integrated in the following way as illustrated in Fig. 8(c). In one layer, we make small disks with radii $r_s - r_c$ that are centered at the central points of triangles (denoted by ⋆’s). We then let the vertices of another layer (denoted by ⋆’s) fall into the small disks and let the corresponding triangle edges in different layers be parallel. The maximal distances between any vertex in the second layer, e.g., node $A$, and the three vertices in the first layer that form a triangle enclosing the vertex in the second layer, e.g., nodes $B$, $C$, and $D$, are less than $r_c$ because the sum of two edges in a triangle is larger than the other.

When $r_c = r_s$, to achieve two-coverage and three-connectivity, we only can put sensors of second layer at the central points of triangles of first layer. This two-covered and three-connected pattern is shown in Fig. 1(c). In this deployment pattern, Voronoi polygons are equilateral triangles with side length $\sqrt{3} r_c$, and the density of this pattern is $d^*$. Anecessary condition of achieving three-connectivity is that there are three neighbors of every sensor, such that the distance between this sensor and each neighbor is not greater than $r_c$. Checking patterns with density $d^*$ among the types from (b) to (l) of congruent Voronoi tessellations listed in Fig. 2 other than the equilateral triangle pattern, no pattern satisfies this condition. Hence, the equilateral triangle pattern is the only $c$-optimal deployment pattern that can achieve two-coverage and three-connectivity.

When $r_c < r_s$ to achieve two-coverage and three-connectivity, for the pattern shown in Fig. 1(c), side length of equilateral triangles in each layer cannot be greater than $\sqrt{3} r_c$. It follows that measure of each Voronoi polygon in this deployment pattern cannot be greater than $3\sqrt{3} r_c^2/4$. Notice the foregoing necessary condition of achieving three-connectivity. Consider the types (b)–(l) of congruent Voronoi tessellations shown in Fig. 2, with an argument similar to that in the proof of Theorem 5.1 and a routine calculating, we can prove that the equilateral triangle pattern is the only $c$-optimal deployment pattern that can achieve two-coverage and three-connectivity.

C. Remarks

We present both one- and two-connected $c$-optimal deployment patterns here since they have the minimal requirement on the $r_c$ range (over $r_s$) for networks to be connected. However, the long-path problem, i.e., a message transmitted between two sensor nodes that are close to each other but on different horizontal lines has to get over a long distance, may exist in some applications using these patterns. To overcome this, we propose three-connected $c$-optimal deployment patterns as above. Note that the connection links in the three-connected deployment patterns are evenly distributed over the network.

Note that when $r_c > \sqrt{3} r_s/2$, there are infinitely many one- or two-connected $c$-optimal deployment patterns. When $r_c > r_s$, there are also infinitely many three-connected $c$-optimal deployment patterns. We can take advantage of the infinite variants of the patterns to meet different design requirements. For example, when three-connectivity and large distances between sensor nodes are needed to improve proper traffic balancing and reduce interference, placing the second layer vertices at the centers of the triangles in the first layer can be a preferred type.

VI. PRACTICAL ISSUES

In reality, the sensing range may not exactly follow the disk sensing model. Megerian et al. [19] propose that sensing quality gradually attenuates with distance. Zhou et al. [29] propose a probabilistic sensing model where the detection probability varies with different target distances. When the above models are used, the sensing disk can be obtained by setting a sensing range threshold that is based on a desirable sensing quality or detection probability. In some cases, the sensing area is non-disk even after a threshold has been set. Cao et al. [6] suggest the sensing capability roughly follows a Gaussian distribution over different directions. Denote the average sensing radius over all $\mu$, and the standard deviation by $\sigma^2$. As $\sigma$ decreases, the sensing field is more like a disk.

We study by simulation the impact from such sensing irregularity on the previously discussed $c$-optimal deployment patterns that are constructed with two layers of triangle lattice patterns. We study three typical cases:

Case 1) when vertices of two triangle lattice layers entirely overlap;

Case 2) when vertices of the second triangle lattice layer are located at the center of the triangles in the first layer, as shown in Fig. 1(c);
Case 3) when triangles in both layers have horizontal edges and the vertices of the second triangle lattice layer are located at the middle of the horizontal edges in the first layer, as shown in Figs. 1(c) or 8(b).

In our simulation, sensor nodes are deployed over a $1000 \times 1000$-m$^2$ region. Each sensor node has a 30-m average sensing range in each direction ($\mu = 30$ m) with variance $\sigma^2$ ranging from 0 to 20. In each simulation, we randomly generate 100 000 points in the deployment region and then check their coverage. We repeat simulations with the same configuration for $\mu$ and $\sigma^2$ 500 times and then take the average.

The respective results are shown in Fig. 10(a)–(c). In these figures, “$k$-coverage” means at least $k$-covered. We observe from the figures that performance under irregularity for all three cases is close when sensing ranges are highly irregular. This indicates that sensor node location information plays an insignificant role at this time. However, there are salient differences when sensing ranges are quite regular, especially when $\sigma^2 < 5$. In cases 2) and 3), there is a higher percentage of areas that are covered by at least three sensors than that in case 1). In case 1), there are more areas that are covered by at least four sensors. Such differences turn out to be negligible as irregularity increases. Interestingly, we observe some convex curves, e.g., the one showing the percentage of four-covered area in case 1). This phenomenon occurs since proper sensor locations with regular sensing disks as well as irregular sensing ranges with not-so-important relative locations can both help achieve a high degree of coverage. The transition between these two yields a temporary decrease of highly covered area.

We also observe that, in practice, the communication model may not follow the disk model. Furthermore, in real-world deployments, deployment errors and dominating geographical constraints of the deployment region must also be taken into account. In some cases, hierarchical sensor network architectures are desirable. Serving as a fundamental reference, our optimal deployment patterns presented in this paper are still essential when these practical issues are addressed. We omit further discussion of these issues in the paper. We refer the interested reader to [1]–[4] and [27] for detailed discussions.

VII. CONCLUSION

WSN deployments with higher degrees of sensing coverage are important. Finding optimal patterns that achieve multiple-coverage with the minimum number of sensor nodes is a fundamental but difficult problem that remains unsolved to date. In this paper, we study $c$-optimal patterns. To the best of our knowledge, this paper makes the first step toward finding the ultimate solutions to the optimal multiple-coverage problem and the optimal connected multiple-coverage problem. Our results have both theoretical and practical significance. Our future work focuses on considering globally optimal multiple-coverage patterns and optimal connected multiple-coverage patterns and extending our patterns to non-disk-based sensing and communication models.

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Ziqiu Yun received the Ph.D. degree in mathematics from Helsinki University, Helsinki, Finland, in 1990. He is a Professor with the Department of Mathematics, Soochow University, Suzhou, China. He is a reviewer of Mathematical Reviews, which is a division of the American Mathematical Society. He was the Vice Chief Director of the Department of Mathematics, Soochow University, from 1995 to 1998. His research interests mainly include general topology, dimension theory, network coverage in WSNs, and rough set theory.

Wei Zhao (F’01) completed the undergraduate program in physics at Shaanxi Normal University, Xi’an, China, in 1977. He received the M.S. and Ph.D. degrees in computer and information sciences from the University of Massachusetts, Amherst, MA, USA, in 1983 and 1986, respectively.

He is currently the Rector of the University of Macau, Macau. Before joining the University of Macau, he served as the Dean of the School of Science, Rensselaer Polytechnic Institute, Troy, NY, USA. Between 2005 and 2006, he served as the director for the Division of Computer and Network Systems, US National Science Foundation (NSF), when he was on leave from Texas A&M University, College Station, TX, USA, where he served as Senior Associate Vice President for Research and Professor of computer science. He was the Founding Director of the Texas A&M Center for Information Security and Assurance, which has been recognized as a Center of Academic Excellence in Information Assurance Education by the National Security Agency. Since 1986, he has served as a faculty member with Amherst College, Amherst, MA, USA; the University of Adelaide, Adelaide, Australia; and Texas A&M University. As an elected IEEE fellow, he has made significant contributions in distributed computing, real-time systems, computer networks, and cyberspace security.

Biao Chen (S’96–M’99) received the B.S. degree in computer science from Fudan University, Shanghai, China, in 1988, and the M.S. degree in mathematics and Ph.D. degree in computer science from Texas A&M University, College Station, TX, USA, in 1992 and 1996, respectively.

After graduation, he joined the Department of Computer Science, University of Texas at Dallas, Dallas, TX, USA, as a faculty member. Currently, he is a Visiting Professor with the Department of Computer and Information Science, University of Macau, Macau. His research interests include distributed systems, networking, and security.

Dr. Chen is a member of Sigma Xi and the Association for Computing Machinery (ACM).

Zuoming Yu received the B.S. and M.S. degrees in electronic engineering from Shanghai Jiao Tong University (SJTU), Shanghai, China, in 1990 and 1993, respectively, and the Ph.D. degree in computer engineering from Texas A&M University, College Station, TX, USA, in 2001.

Currently, he is an Associate Professor with the Department of Computer Science and Engineering, The Ohio State University (OSU), Columbus, OH, USA. His research interests include distributed computing, computer networks, and cyberspace security.

Dr. Xuan is a member of the Association for Computing Machinery (ACM). He received the National Science Foundation (NSF) CAREER Award in 2005 and the College of Engineering/OSU Lumley Research Award in 2009.

Jin Teng (S’09) received the B.S and M.S degrees in electronic engineering from Shanghai Jiao Tong University, Shanghai, China, in 2006 and 2009, respectively, and is currently pursuing the Ph.D. degree in computer science and engineering at The Ohio State University, Columbus, OH, USA.

His research interests mainly include wireless communication architecture, QoS of wireless networks, network coverage in WSNs, and cyberspace security.

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