On Multiterminal Source Code Design

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Abstract—Multiterminal (MT) source coding refers to separate lossy encoding and joint decoding of multiple correlated sources. Recently, the rate region of both direct and indirect MT source coding in the quadratic Gaussian setup with two encoders was determined. We are thus motivated to design practical MT source codes that can potentially achieve the entire rate region. In this paper, we present two practical MT coding schemes under the framework of Slepian–Wolf coded quantization (SWCQ) for both direct and indirect MT problems. The first, asymmetric SWCQ scheme relies on quantization and Wyner–Ziv coding, and it is implemented via source splitting to achieve any point on the sum-rate bound. In the second, conceptually simpler scheme, symmetric SWCQ, the two quantized sources are compressed using symmetric Slepian–Wolf coding via a channel code partitioning technique that is capable of achieving any point on the Slepian–Wolf sum-rate bound. Our practical designs employ trellis-coded quantization and turbo/low-density parity-check (LDPC) codes for both asymmetric and symmetric Slepian–Wolf coding. Simulation results show a gap of only 0.139–0.194 bit per sample away from the sum-rate bound for both direct and indirect MT coding problems.

Index Terms—CEO problem, multiterminal (MT) source coding, Slepian–Wolf coded quantization (SWCQ), Slepian–Wolf (SW) coding, trellis-coded quantization, Wyner–Ziv (WZ) coding.

I. INTRODUCTION

In many emerging applications (e.g., distributed sensor networks), multiple correlated sources need to be separately compressed at distributed terminals and transmitted to a central unit. Due to complexity and power constraints, the transmitters are often not allowed to communicate with each other. This gives rise to the problem of multiterminal source coding [4], which has 30 years of history.

Multiterminal (MT) source coding is a distributed source coding problem. Distributed source coding was started by Slepian and Wolf in 1973 [32], who considered separate lossless compression of two correlated sources, and showed the surprising result that separate encoding and joint decoding suffer no rate loss compared to the case when the sources are compressed jointly. Their seminal work [32] was subsequently extended to other distributed source coding scenarios. In 1976, Wyner and Ziv [40] extended one special case of Slepian–Wolf (SW) coding, namely, lossless source coding with decoder side information, to lossy source coding with decoder side information. Unlike SW coding, there is in general a rate loss with Wyner-Ziv (WZ) coding [40] compared to the lossy source coding problem when side information is also available at the encoder. An exception occurs when the source and side information are jointly Gaussian and the distortion measure is mean-squared error (MSE).

Soon after the celebrated works of Slepian and Wolf [32] and Wyner and Ziv [40], Berger [4] introduced the general problem of MT source coding by considering a more general case of separate lossy source coding of two (or more) sources. Two classes of MT source coding problems have been studied in the literature. In the original work of Berger and Tung [4], [35], the case where each encoder observes directly its source was considered; later, Yamamoto and Itoh [42] and Flynn and Gray [11] focused on another scenario where each encoder cannot observe directly the source that is to be reconstructed at the decoder, but is rather provided only with a noisy version. These two classes are distinguished as the direct and indirect (or remote) MT source coding problem, respectively. Note that in the latter case, often referred to as the CEO problem [23], [37], a single source is to be reconstructed at the decoder.

Theoretical study of the MT source coding problem amounts to determining the achievable rate region (i.e., all possible compression rate tuples) under distortion constraint(s) on the source(s). Finding the achievable rate region for general MT source coding is a difficult task and still remains open. Only inner and outer bounds for both MT coding problems have been provided [4], [11], [35], [42].

Owing to the difficulty of the general MT source coding problem, researchers have focused on the quadratic Gaussian setup with Gaussian source(s) and MSE distortion measure. Theoretical results on the quadratic Gaussian MT source coding problem appeared in [4], [22], [35] for the direct setting and in [6], [23], [24], [26], [37] for the indirect/CEO setting. However, even for this special case, the achievable rate region was unknown until recently. The indirect/CEO problem (with arbitrary number of encoders) was solved independently by Oohama [24] in 1999 (and published recently in [25]) and Prabhakaran et al.

1One can loosely think of MT source coding as the lossy version of SW coding.

2All rate points within the inner bound are achievable, while those outside the outer bound are not.
[26], using the entropy power inequality [9]. But the direct MT source coding problem is more challenging because it requires the reconstruction of a vector source instead of a single remote source, and the lack of a vector version of the entropy power inequality has prevented the generalization of the proofs of [25], [26]. Consequently, the exact achievable rate region is still unknown for the direct MT source coding problem with arbitrary number of encoders. However, for the case with two encoders, Wagner et al. [38] made the connection in 2005 between the direct and indirect MT source coding problems (via a so-called $\mu$-sum problem) and showed tightness of the Berger–Tung achievable bound [4], [35] by proving the converse.

With the precise rate regions for both the direct and indirect quadratic Gaussian MT problems with two encoders recently provided in [25], [26], [38], now is the time to study practical code designs that are capable of achieving any point in these regions. Compared to the body of theoretical works on MT source coding problems, research on practical code designs is still in its infancy. Targeting the tight sum–rate bound for the two-encoder quadratic Gaussian CEO problem [25], [26], Pradhan and Ramchandran [28] provided a code design based on generalized coset codes, with fixed-rate scalar quantizers and trellis codes. Although capable of trading off transmission rates between the two encoders, the design in [28] performs relatively far away from the theoretical limits, especially at low rates. Motivated by the fact that WZ coding [40] is a special case of MT coding, in an earlier work [44], we proposed an asymmetric coding system for the CEO problem that essentially relies on WZ coding. Although the scheme in [44] gives better results than those of [28], it is limited to approaching the two corner points of the achievable region only.

In this paper, we focus on practical code designs for the quadratic Gaussian direct and indirect MT problems with two encoders. Generally speaking, MT source coding is a joint source–channel coding problem: first, its lossy nature necessitates quantization of the sources; second, the distributed nature of the encoders calls for compression (after quantization) by SW coding, which is commonly implemented by a channel code. More importantly, one of the conclusions of the theoretical works of [25], [26], [38] is that vector quantization (VQ) plus SW coding is indeed optimal for the quadratic Gaussian MT source coding with two terminals. Following this guiding principle, we propose a framework called Slepian–Wolf coded quantization (SWCQ) for practical MT source coding. Unlike nested lattice codes suggested by Zamir et al. [50] and generalized coset codes used by Pradhan and Ramchandran [28], which are essentially nested source–channel codes, SWCQ explicitly separates the SW coding component from the vector quantizers at the encoder (while employing joint estimation/reconstruction at the decoder). This approach not only allows us to design a good source code and a good channel code individually, but also enables us to evaluate the practical performance loss due to source coding and channel coding separately. Moreover, SWCQ is very general as it applies to both direct and indirect MT source coding problems. It also generalizes similar approaches recently developed in [17], [43] for WZ coding.

Slepian and Wolf [32] showed that the separate compression of two correlated sources can be near lossless at the total rate of their joint entropy. In particular, when one source is available only at the decoder as side information, the other source can still be near-losslessly compressed at the rate of its conditional entropy given the decoder side information. This special case corresponds to the two corner points of the SW rate region, and is called asymmetric SW coding; on the other hand, symmetric (or more precisely, nonasymmetric) SW coding attempts to approach any point between the two corner points. Correspondingly, two classes of SW code designs exist in the literature. Asymmetric SW code designs based on coset codes [27], turbo codes [1], [2], [13], [19], and low-density parity-check (LDPC) codes [18], [34] were developed for binary sources. The main idea [39] is to compress a binary input source sequence to the syndrome of a linear channel code for the “virtual” correlation channel between the source and the decoder side information, and find the binary sequence with the same syndrome that is closest to the side information at the decoder. This syndrome-based method can approach one of the two corner points of the SW rate region if the employed channel code approaches the capacity of the “virtual” correlation channel.

In practical applications (e.g., sensor networks), it is preferable for the encoders to be able to operate at flexible rates. This necessitates symmetric SW coding. The most straightforward approach is time-sharing between the two corner points. However, time-sharing might not be practical because it requires synchronization between the encoders. An alternative is the source splitting approach introduced by Rimoldi and Urbanke [30]. By “splitting” one source into two subsources, arbitrary point on the two-terminal SW rate region can be mapped to the corner point of a three-terminal SW rate region, which can be approached using asymmetric SW coding. A drawback of source splitting is that it increases coding complexity and introduces extra error propagations. Recently, Pradhan and Ramchandran [28] suggested a method for symmetric SW coding based on partitioning a single parity-check code. Following this idea, in [31], a practical code design method for symmetric SW coding of uniform binary sources was developed; assuming binary symmetric correlation channel between two sources, the designs of [31] with irregular repeat–accumulate codes [16] and turbo codes [5] give results that are very close to the SW limit.

Combining trellis-coded quantization (TCQ) [21], as the most powerful source coding technique, with asymmetric and symmetric SW coding, respectively, we present in this paper two practical designs under the SWCQ framework for both direct and indirect quadratic Gaussian MT source coding with two encoders. The first asymmetric SWCQ scheme employs quantization (i.e., TCQ), asymmetric SW coding, and source splitting to realize MT source coding with two encoders. More precisely, our MT source code design is “split” into one classic source coding component and two WZ coding components. While classic source coding relies on entropy-coded VQ, WZ coding is implemented by combining TCQ and turbo/LDPC codes (for asymmetric SW coding).

In our second symmetric SWCQ scheme, the outputs of two TCQs are compressed using symmetric SW coding, which is
based on the concept of channel code partitioning [31] for arbitrary rate allocation between the two encoders. Exploiting the joint statistics of the quantized sources, we develop a multilevel channel coding framework for symmetric SW coding. Furthermore, arithmetic coding [3] is employed at each encoder to exploit the cross-bit-plane correlation in each of the quantized sources for further compression.

To demonstrate the effectiveness of our proposed SWCQ framework, we show that, assuming ideal source coding and ideal SW coding (realized, for example, via capacity-achieving channel coding), both asymmetric SWCQ and symmetric SWCQ can achieve any point on the sum–rate bound of the rate region for both direct and indirect MT source coding. We also perform high-rate performance analysis of SWCQ under practical TCQ and ideal SW coding. Practical designs using TCQ and turbo/LDPC codes for asymmetric SW coding, and TCQ, arithmetic coding, and turbo/LDPC code for symmetric SW coding perform only 0.139–0.194 bit per sample (b/s) away from the sum–rate bounds of quadratic Gaussian MT source coding.

In summary, the main contributions of this paper are as follows.

1) The SWCQ framework based on separate vector quantization and SW coding for the quadratic Gaussian direct and indirect MT source coding problems with two encoders.
2) Demonstration of optimality of SWCQ for quadratic Gaussian MT source coding in the sense of being able to approach arbitrary points on the sum–rate bounds, assuming ideal source coding and ideal SW coding.
3) High-rate performance analysis of SWCQ for MT source coding under practical TCQ and ideal SW coding.
4) Characterization of the joint behavior of two independently dithered TCQ quantizers with independent and identically distributed (i.i.d.) dither sequences; the quantization noises of the two quantizers are shown to be (nearly) independent, which is required by optimality of an MT source coding scheme.
5) An efficient multilevel symmetric SW code design that extends channel code partitioning approach for binary sources [31] to arbitrary correlation models among the sources; this design is capable of exploiting the joint statistics of the quantization indices and incorporating the statistics into the decoding algorithm.
6) Practical asymmetric and symmetric MT code designs with dithered TCQ and multilevel asymmetric/symmetric SW coding that come much closer to the sum–rate bounds of direct and indirect MT problems with two encoders than the design of [28].

The rest of the paper is organized as follows. Section II gives the formal definitions of the direct and indirect MT source coding problems and reviews the theoretical bounds for both the general and quadratic Gaussian settings. Section III puts forth the framework of SWCQ for MT source coding and shows its optimality under ideal quantization and ideal SW coding. Section IV gives details on practical quantizer design of our practical SWCQ schemes and provides high-rate performance of SWCQ under practical TCQ and ideal SW coding. Section V describes asymmetric and symmetric SW code designs based on turbo/LDPC codes for MT source coding. Section VI presents simulation results, and Section VII concludes the paper.

Notation-wise, random variables are denoted by capital letters, e.g., $X$. They take values $x$ from alphabet $\mathcal{X}$. Random vectors are denoted by capital letters superscripted by their lengths, e.g., $X^n$. All channel codes are binary. Matrices are denoted by bold-face upper-case letters. $I_k$ is the $k \times k$ identity matrix and $O_{k_1 \times k_2}$ the $k_1 \times k_2$ all-zero matrix. All logarithms are of base two unless otherwise specified.

II. THEORETICAL LIMITS OF MT SOURCE CODING

In this section, we review theoretical bounds of direct and indirect MT source coding.

A. Direct MT Source Coding

The direct MT source coding setup is depicted in Fig. 1. The encoders observe sources $Y_1$ and $Y_2$, which take values in $\mathcal{Y}_1 \times \mathcal{Y}_2$, and are drawn i.i.d. from the joint probability density function (pdf) $f_{Y_1,Y_2}(y_1,y_2)$. Each sequence of $n$ source samples is grouped as a source block $Y_1^n$ and $Y_2^n$, where $Y_1^n = \{Y_{1,i}\}_{i=1}^n$, $Y_2^n = \{Y_{2,i}\}_{i=1}^n$. Two encoder functions

$$\phi_1 : \mathcal{Y}_1^n \rightarrow \{1,2,\ldots,2^{nR_1}\}$$
$$\phi_2 : \mathcal{Y}_2^n \rightarrow \{1,2,\ldots,2^{nR_2}\}$$

(1)

separately compress $Y_1^n$ and $Y_2^n$ to $W_1$ and $W_2$ at rates $R_1$ and $R_2$, respectively. A decoder function

$$\varphi : \{1,2,\ldots,2^{nR_1}\} \times \{1,2,\ldots,2^{nR_2}\} \rightarrow \mathcal{Y}_1^n \times \mathcal{Y}_2^n$$

(2)

reconstructs the source block as $\{\hat{Y}_{1,i},\hat{Y}_{2,i}\}$ based on the received $W_1$ and $W_2$.

For a distortion pair $(D_1,D_2)$ and a given distortion measure $d(\cdot,\cdot)$, a rate pair $(R_1,R_2)$ is achievable if for any $\epsilon > 0$, there exists a large enough $n$ and a triple $(\phi_1,\phi_2,\varphi)$ such that the distortion constraints

\[ \frac{1}{n} \sum_{i=1}^n E[d(Y_{1,i},\hat{Y}_{1,i})] \leq D_1 + \epsilon \]
\[ \frac{1}{n} \sum_{i=1}^n E[d(Y_{2,i},\hat{Y}_{2,i})] \leq D_2 + \epsilon \]

(3)

are satisfied. The achievable rate region $\mathcal{R}^*(D_1,D_2)$ is the convex hull of the set of all achievable rate pairs $(R_1,R_2)$.
The exact achievable rate region for the direct MT source coding problem is still unknown. Only inner and outer rate regions are provided. For auxiliary random variables \( Z_1 \) and \( Z_2 \) let
\[
\hat{R}(Z_1, Z_2) = \{(R_1, R_2) : R_i \geq I(Y_i Y_{2i}; Z_i|Z_j), \quad i, j = 1, 2, i \neq j, R_1 + R_2 \geq I(Y_1 Y_2; Z_1 Z_2)\}; \quad (4)
\]
then the inner rate region is given by \([4], [35], [42]\) as shown in (5) at the bottom of the page, while the outer rate region is \([4], [35], [42]\) as shown in (6) at the bottom of the page, where \( \text{conv}(\cdot) \) represents convex closure. Let \( \partial\hat{R}(D_1, D_2) \) be the set of all boundary points of the rate region \( \hat{R}(D_1, D_2) \); likewise, let \( \partial\hat{R}(D_1, D_2) \) be the set of all boundary points of the rate region \( \hat{R}(D_1, D_2) \). We call \( \partial\hat{R}(D_1, D_2) \) the inner bound and \( \partial\hat{R}(D_1, D_2) \) the outer bound.

For the direct Gaussian MT source coding problem with MSE distortion measure \( d(x, \cdot) \), where the sources \( (Y_1, Y_2) \) are jointly Gaussian random variables with variances \( (\sigma^2_{y_1}, \sigma^2_{y_2}) \) and correlation coefficient \( \rho = \frac{E[Y_1 Y_2]}{\sigma_{y_1}\sigma_{y_2}} \), the Berger–Tung (BT) inner rate region (5) becomes [22]
\[
\hat{R}^{\text{BT}}(D_1, D_2) = \hat{R}^{\text{BT}}_1(D_1, D_2) \cap \hat{R}^{\text{BT}}_2(D_1, D_2) \cap \hat{R}^{\text{BT}}_{12}(D_1, D_2) \quad (7)
\]
where
\[
\hat{R}^{\text{BT}}_i(D_1, D_2) = \left\{(R_1, R_2) : R_i \geq \frac{1}{2} \log^+ \left[\frac{(1 - \rho^2 + \rho^22^{-2R_i})\sigma_{y_i}^2}{D_i}\right], \quad i, j = 1, 2, i \neq j\right\}, \quad (8)
\]
\[
\hat{R}^{\text{BT}}_{12}(D_1, D_2) = \left\{(R_1, R_2) : R_1 + R_2 \geq \frac{1}{2} \log^+ \left[\frac{(1 - \rho^2)\beta_{\text{max}}\sigma_{y_1}^2\sigma_{y_2}^2}{2D_1D_2}\right]\right\}, \quad (9)
\]
with
\[
\beta_{\text{max}} = 1 + \sqrt{1 + \frac{4\rho^2D_1D_2}{(1 - \rho^2)^2\sigma_{y_1}^2\sigma_{y_2}^2}}
\]
and \( \log^+ x = \max\{\log x, 0\} \).

Recently, the achievable BT rate region \( \hat{R}^{\text{BT}}(D_1, D_2) \) has been shown to be tight [38] for the two-terminal direct Gaussian MT source coding problem, that is, \( \hat{R}^{\text{BT}}(D_1, D_2) = \hat{R}^\star(D_1, D_2) \). The boundary of the rate region \( \hat{R}^{\text{BT}}(D_1, D_2) \)
\[
\hat{R}(D_1, D_2) = \text{conv}\{\hat{R}(Z_1, Z_2) : Z_1 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Z_2, \exists \varphi(Z_1^\prime, Z_2^\prime) \text{ satisfying (3)}\} \quad (5)
\]
\[
\hat{R}(D_1, D_2) = \text{conv}\{\hat{R}(Z_1, Z_2) : Z_1 \rightarrow Y_1 \rightarrow Y_2, Z_2 \rightarrow Y_2 \rightarrow Y_1, \exists \varphi(Z_1^\prime, Z_2^\prime) \text{ satisfying (3)}\}. \quad (6)
\]

Fig. 2. The BT rate region for the direct Gaussian MT source coding problem with \( \sigma^2_{y_1} = \sigma^2_{y_2} = \sigma^2 = 1, \rho = 0.9, D_1 = D_2 = 0.1 \).
B. Indirect MT Source Coding

The indirect MT source coding setup with two encoders is depicted in Fig. 3. The remote source $X$ and two noises $N_1$ and $N_2$ are mutually independent i.i.d. random variables drawn from the joint pdf $f_{X,N_1,N_2}(x,n_1,n_2) = f_X(x)f_{N_1}(n_1)f_{N_2}(n_2)$. The block $\{Y_1^n, Y_2^n\}$ is a length-$n$ sequence of noisy observations: $Y_1^n = X^n + N_1^n, Y_2^n = X^n + N_2^n$ at the two encoders. The indirect system shares the form of encoder functions $(\phi_1, \phi_2)$ with the direct system (1), while having a different decoder function $\psi: \{1,2,\ldots,2^{nR_1}\} \times \{1,2,\ldots,2^{nR_2}\} \rightarrow X^n$ which reconstructs the remote source block as $\hat{X}^n$. Similar to the direct case, we define the achievable rate region $\mathcal{R}^*(D)$ as the convex hull of the set of all achievable rate pairs $(R_1, R_2)$ such that for any $\epsilon > 0$, there exists a large enough $n$ and a triple $(\phi_1, \phi_2, \psi)$ satisfying the distortion constraint

$$\frac{1}{n} \sum_{i=1}^{n} E[|d(X_i, \hat{X}_i)|] \leq D + \epsilon.$$  

(12)

The exact achievable rate region for the indirect MT source coding problem is also unknown. For auxiliary random variables $Z_1$ and $Z_2$, the inner rate region is given by [4], [35], [42] as shown in (14) at the bottom of the page, while the outer rate region is [4], [35], [42] are given in (15), also at the bottom of the page.

In the indirect Gaussian MT source coding problem with MSE distortion measure, $X$ is an i.i.d. Gaussian random variable $\sim \mathcal{N}(0, \sigma_x^2)$, and for $i = 1, 2$ the noisy observations at the two encoders are given by $Y_i = X + N_i$, where $N_1 \sim \mathcal{N}(0, \sigma_{n_1}^2)$ and $N_2 \sim \mathcal{N}(0, \sigma_{n_2}^2)$ are i.i.d. Gaussian random variables independent of each other and $X$. For this special case, Yamamoto and Itoh [42] reported the Yamamoto–Itoh (YI) achievable rate region, which can be expressed in an equivalent form in terms of $(\sigma_x^2, \sigma_{n_1}^2, \sigma_{n_2}^2, D)$ as

$$\hat{\mathcal{R}}_{\psi}^Y(D) = \text{conv} \left( \hat{\mathcal{R}}_1^Y(D) \cap \hat{\mathcal{R}}_2^Y(D) \cap \hat{\mathcal{R}}_{12}^Y(D) \right)$$

(16)

where $\hat{\mathcal{R}}_1^Y(D)$ and $\hat{\mathcal{R}}_2^Y(D)$ are given in (17)–(18) at the bottom of the page.

The YI achievable rate region (16) is shown to be tight [25], [26], that is, $\hat{\mathcal{R}}_1^Y(D) = \mathcal{R}^*(D)$. The boundary of $\hat{\mathcal{R}}_1^Y(D)$ consists of a diagonal line segment and two curved portions (see Fig. 4 for an example) iff

$$\hat{\mathcal{R}}(D) = \text{conv} \{ \hat{\mathcal{R}}(Z_1, Z_2): Z_1 \rightarrow Y_1 \rightarrow X \rightarrow Y_2 \rightarrow Z_2, \exists \psi(Z_1^n, Z_2^n) \text{ satisfying (13)} \}$$

(14)

$$\hat{\mathcal{R}}(D) = \text{conv} \{ \hat{\mathcal{R}}(Z_1, Z_2): Z_1 \rightarrow Y_1 \rightarrow X \rightarrow Y_2 \rightarrow Z_2 \rightarrow Y_2 \rightarrow X \rightarrow Y_1, \exists \psi(Z_1^n, Z_2^n) \text{ satisfying (13)} \}.$$  

(15)

$$\hat{\mathcal{R}}_i^Y(D) = \left\{ (R_1, R_2): R_i \geq \frac{1}{2} \log^+ \left[ \frac{\sigma_x^2(\sigma_x^2 + 2R_j \sigma_{n_j}^2 + \sigma_{n_j}^2)^2(\sigma_x^2 + \sigma_{n_j}^2)^{-1}}{2 - 2R_j \sigma_x^2(D - \sigma_{n_j}^2) + \sigma_x^2 D(\sigma_{n_j}^2 + \sigma_{n_j}^2) - \sigma_x^2 \sigma_{n_j}^2(\sigma_x^2 - D)} \right] \right\}, \quad i = 1, 2, i \neq j$$

(17)

$$\hat{\mathcal{R}}_{12}^Y(D) = \left\{ (R_1, R_2): R_1 + R_2 \geq \frac{1}{2} \log^+ \left[ \frac{4\sigma_x^2}{\sigma_{n_1}^2 \sigma_{n_2}^2 (\frac{1}{(\sigma_x^2 - D)} + \frac{1}{\sigma_{n_1}^2} + \frac{1}{\sigma_{n_2}^2})} \right] \right\},$$

(18)
Under this constraint, the sum-rate bound \( \partial R_{12}(D) \) is defined as the set of all achievable rate pairs that minimize the sum-rate \( R = R_1 + R_2 \).

Note that in the symmetric case with \( \sigma_i^2 = \sigma_2^2 = \sigma_n^2 \), the sum-rate bound \( \partial R_{12}(D) \) becomes

\[
\partial R_{12}(D) = \left\{ (R_1, R_2) : R_1 + R_2 = \frac{1}{2} \log^+ \left[ \frac{2 \sigma_i^2}{(\sigma_n^2 + D) \theta} \right] \right\}
\]

where \( \theta = 1 - \frac{2 \sigma_i^2}{(\sigma_n^2 + D)} \).

III. CODE DESIGN FOR MT SOURCE CODING

In this section, we propose two code designs for the direct and indirect Gaussian MT coding problems, which are capable of trading off rates between the two encoders. The first is based on asymmetric SWCQ, which employs quantization and asymmetric SW coding, and is implemented via source splitting [30]. The second relies on symmetric SWCQ, which exploits quantization and symmetric SW coding [31]. We show that using random binning argument [9], both designs can potentially approach any point on the sum-rate bound in either of the two Gaussian MT coding problems.

A. Asymmetric SWCQ

Asymmetric SWCQ is schematically depicted in Fig. 5 in conjunction with source splitting for MT source coding. It consists of a classical source encoder/decoder pair, two WZ encoder/decoder pairs, and a linear combinator.

The classical source encoder/decoder pair is defined by the following four functions:

Encoder:

\[
Q_{1} : \mathbb{Y}_1^n \rightarrow \{1, 2, \ldots, 2^n R_1^n \} \\
E_{ASW} : \{1, 2, \ldots, 2^n R_1^n \} \rightarrow \{1, 2, \ldots, 2^n R_2^n \} \\
D_{ASW} : \{1, 2, \ldots, 2^n R_2^n \} \rightarrow \{1, 2, \ldots, 2^n R_1^n \} \\
Q_{21}^{-1} : \{1, 2, \ldots, 2^n R_2^n \} \rightarrow \mathbb{Z}_{21}^{n}
\]

where \( R_1^n \) is the quantization rate of Quantizer I, \( R_2^n \) is the transmission rate of the classical source encoder, and \( \mathbb{Z}_{21}^{n} \) is an \( n \)-dimensional vector codebook of size \( 2^{n R_2} \). Quantizer I first quantizes \( \mathbb{Y}_1^n \) (which is a block of \( n \) source samples in the direct or a block of noisy observations in the indirect setup) using codebook \( \mathbb{Z}_1^n \) by finding the vector codeword \( \mathbb{Z}_1^n \) that is “closest” (e.g., in Euclidean distance) to \( \mathbb{Y}_1^n \), and outputs the quantization index \( I_1 = Q_1(\mathbb{Y}_1^n) = i_1(\mathbb{Y}_1^n) \). Then the entropy encoder compresses \( I_1 \) to \( S_1 = E_{\text{ASW}}(I_1) \), which is transmitted at rate \( R_1 \) b/s. At the decoder side, the classical source decoder losslessly decompresses \( S_1 \) to \( I_1 = D_{\text{ASW}}(S_1) \) using the entropy decoder, and then employs Dequantizer II to reconstruct \( Z_1^n \) as \( Z_1^n = Q_1^{-1}(I_1) \). Operations in the classical source encoder/decoder pair can be summarized as

Encoder: \( S_1 = E_{\text{ASW}}[Q_1(\mathbb{Y}_1^n)] \)

Decoder: \( \hat{Z}_1^n = Q_1^{-1}[D_{\text{ASW}}(S_1, Z_{21}^{n})] \).

(21)
To generate the side information for the second WZ encoder/decoder pair, the linear combinator \( \psi_c : \mathcal{Z}_1^{n} \times \mathcal{Z}_2^{n} \to \mathcal{Z}_c^{n} \) implements a linear function \( \tilde{Z}_c^n = \psi_c(\tilde{Z}_1^n, \tilde{Z}_2^n) = \alpha_c \tilde{Z}_1^n + \beta_c \tilde{Z}_2^n \).

WZ encoder/decoder pair II then implements the following four functions:

\[
\begin{align*}
Q_{22} & : \mathcal{Y}_2^n \to \{1, 2, \ldots, 2^n R_{22}^b\} \\
E_{22}^{ASW} & : \{1, 2, \ldots, 2^n R_{22}^b\} \to \{1, 2, \ldots, 2^n R_{22}^r\} \\
D_{22}^{SW} & : \{1, 2, \ldots, 2^n R_{22}^r\} \times S_{22} \to \mathcal{Z}_2^n \\
\tilde{Q}_{22}^{-1} & : \{1, 2, \ldots, 2^n R_{22}^b\} \to \mathcal{Z}_2^n
\end{align*}
\]

where \( R_{22}^b \) is the quantization rate of Quantizer III, \( R_{22}^r \) the transmission rate of the WZ Encoder II, and \( \mathcal{Z}_2^n \) a codebook of size \( 2^n R_{22}^b \), which is used in Quantizer III to quantize \( Y_2^n \). The resulting quantization index \( I_{22} = Q_{22}(Y_2^n) \triangleq i_{22}(Z_2^n) \) is compressed by Asymmetric SW Encoder II to \( S_{22} = E_{22}^{ASW}(I_{22}) \), which is transmitted at rate \( R_{22}^b \) b/s. With \( \tilde{Z}_2^n \) as side information, WZ Decoder II generates \( \tilde{I}_{22} = D_{22}^{SW}(S_{22}, \tilde{Z}_2^n) \) as the reconstruction of \( I_{22} \), and decodes it to \( \tilde{Z}_2^n = \tilde{Q}_{22}(\tilde{I}_{22}) \triangleq i_{22}(I_{22}) \) with Dequantizer III. Operations in the linear combinator and the WZ encoder/decoder pair II can be summarized as

\[
\text{Encoder: } S_{22} = E_{22}^{ASW}
\begin{bmatrix}
Q_{22}(Y_2^n) \\
\end{bmatrix}
\]

\[
\text{Decoder: } \tilde{Z}_{22} = \tilde{Q}_{22}^{ASW}
\begin{bmatrix}
D_{22}^{SW}(S_{22}, \psi_c(\tilde{Z}_1^n, \tilde{Z}_2^n)) \\
\end{bmatrix}
\]  

(23)

Note that, Encoder I and Encoder II separately encode \( Y_1^n \) and \( Y_2^n \) using rates \( R_1^b \) b/s and \( R_2^b \triangleq R_1^b + R_{22}^b \) b/s, respectively; decoder then reconstructs the three quantized versions of the sources as \( \{\tilde{Z}_1^n, \tilde{Z}_2^n, \tilde{Z}_n^{22}\} \).

Our design for direct MT coding is a combination of asymmetric SWCQ and linear estimator, which implements the function \( \psi_{\text{direct}}^{\text{ASWCQ}} : \mathcal{Z}_1^n \times \mathcal{Z}_2^n \times \mathcal{Z}_n^{22} \to \mathcal{Y}_1^n \times \mathcal{Y}_2^n \) defined by

\[
\left( \frac{Y_1^n}{Y_2^n} \right) = \left( \begin{array}{c}
\alpha_1^{A_1} \beta_1^{A_2} \\
\alpha_2^{A_1} \beta_2^{A_2} \\
\end{array} \right) \left( \begin{array}{c}
\tilde{Z}_1^n \\
\tilde{Z}_2^n \\
\tilde{Z}_n^{22} \\
\end{array} \right)
\]

(24)

Similarly, our design for indirect MT coding is a combination of asymmetric SWCQ and linear estimator, which implements the function \( \psi_{\text{indirect}}^{\text{ASWCQ}} : \mathcal{Z}_1^n \times \mathcal{Z}_2^n \times \mathcal{Z}_n^{22} \to \mathcal{X}^n \) defined by

\[
\hat{X}^n = (\alpha_x^{A_1} \beta_x^{A_2} \gamma_x^{A_3}) \left( \begin{array}{c}
\tilde{Z}_1^n \\
\tilde{Z}_2^n \\
\tilde{Z}_n^{22} \\
\end{array} \right)^T.
\]

(25)

The following two theorems state that our asymmetric SWCQ designs can approach any point on the sum-rate bound \( \partial R_{12}^{BT}(D_1^*, D_2^*) \) in the direct MT setting and \( \partial R_{12}^{Y_1}(D_1^*, D_2^*) \) in the indirect setting.

**Theorem 1:** Let \( (R_1^*, R_2^*) \) be any point on the sum-rate bound \( \partial R_{12}^{BT}(D_1^*, D_2^*) \) of (9) for the direct MT problem (assume (10) is satisfied). For any \( \epsilon > 0 \), there exists a block length \( n \), two asymmetric SWCQ encoders \( E_1, E_2 \), which separately compress sources \( Y_1 \) and \( Y_2 \) at rates \( R_1 \) and \( R_2 \), respectively, and an asymmetric decoder \( D \), which jointly reconstructs the sources as \( \hat{Y}_1 \) and \( \hat{Y}_2 \), such that

\[
\frac{1}{n} \sum_{i=1}^{n} E[(Y_{j,i} - \hat{Y}_{j,i})^2] < D_j^* + \epsilon, \quad j = 1, 2
\]

(26)

\[
R_j < R_j^* + \epsilon, \quad j = 1, 2
\]

(27)

*Proof:* See Appendix A. \( \square \)

**Theorem 2:** Let \( (R_1^*, R_2^*) \) be any point on the sum-rate bound \( \partial R_{12}^{Y_1}(D_1^*, D_2^*) \) of (18) for the indirect MT problem (assume (19) is satisfied). For any \( \epsilon > 0 \), there exists a block length \( n \), two asymmetric SWCQ encoders \( E_1, E_2 \), which separately compress observations \( Y_1 \) and \( Y_2 \) at rates \( R_1 \) and \( R_2 \), respectively, and an asymmetric decoder \( D \), which reconstructs source \( X \) as \( \hat{X} \), such that

\[
\frac{1}{n} \sum_{i=1}^{n} E[(X_i - \hat{X}_i)^2] < D^* + \epsilon
\]

(28)

\[
R_j < R_j^* + \epsilon, \quad j = 1, 2
\]

(29)

*Proof:* See Appendix B. \( \square \)

**B. Symmetric SWCQ**

Symmetric SWCQ is schematically depicted in Fig. 6. **Quantizer I** and **Quantizer II** separately quantize \( Y_1^n \) and \( Y_2^n \) using \( n \)-dimensional codebooks \( \mathcal{Z}_1^n \) and \( \mathcal{Z}_2^n \) of size \( 2^n R_1^b \) and \( 2^n R_2^b \), respectively. The resulting quantization indices \( I_1 = Q_1(Y_1^n) \triangleq i_1(Z_1^n) \) and \( I_2 = Q_2(Y_2^n) \triangleq i_2(Z_2^n) \) are separately compressed by Symmetric SW Encoder I

\[
E_1^{\text{SSW}} : \{1, 2, \ldots, 2^n R_1^b\} \to \{1, 2, \ldots, 2^n R_1^r\}
\]

and Symmetric SW Encoder II

\[
E_2^{\text{SSW}} : \{1, 2, \ldots, 2^n R_2^b\} \to \{1, 2, \ldots, 2^n R_2^r\}
\]

defined by \( S_1 = E_1^{\text{SSW}}(I_1) \) and \( S_2 = E_2^{\text{SSW}}(I_2) \), respectively. The transmission rates for the two encoders are \( R_1^b \) b/s and \( R_2^b \) b/s, respectively.

At the decoder side, the Symmetric SW decoder jointly reconstructs the quantization indices \( I_1 \) and \( I_2 \) based on the received messages \( S_1 \) and \( S_2 \). Specifically, it implements a function

\[
D^{\text{SSW}} : \{1, 2, \ldots, 2^n R_1^b\} \times \{1, 2, \ldots, 2^n R_2^b\} \to \{1, 2, \ldots, 2^n R_1^r\} \times \{1, 2, \ldots, 2^n R_2^r\}
\]

(30)
defined by \((\hat{I}_1, \hat{I}_2) = D_{SWW}(S_1, S_2)\). Finally, Dequantizer I and Dequantizer II reproduce the codewords as \(Z^n_1 = Q_1^{-1}(\hat{I}_1) \triangleq \hat{z}_1^n(\hat{I}_1)\) and \(Z^n_2 = Q_2^{-1}(\hat{I}_2) \triangleq \hat{z}_2^n(\hat{I}_2)\), respectively.

Our direct MT code design is a combination of symmetric SWCQ and linear estimator, which implements the function \(\psi_{\text{direct}} : \mathcal{Z}_1^n \times \mathcal{Z}_2^n \rightarrow Y_1^n \times Y_2^n\) defined by

\[
\begin{pmatrix}
Y_1^n \\
Y_2^n
\end{pmatrix} = \begin{pmatrix}
\alpha_S & \beta_S \\
\alpha_S & \beta_S
\end{pmatrix}
\begin{pmatrix}
Z_1^n \\
Z_2^n
\end{pmatrix}.
\]

Similarly, our indirect MT code design is a combination of symmetric SWCQ and linear estimator, which implements the function \(\psi_{\text{indirect}} : \mathcal{Z}_1^n \times \mathcal{Z}_2^n \rightarrow X^n\) defined by

\[
X^n = \begin{pmatrix}
\alpha_S & \beta_S
\end{pmatrix}
\begin{pmatrix}
Z_1^n \\
Z_2^n
\end{pmatrix}.
\]

Similar to the asymmetric SWCQ scheme, the following two theorems assert optimality of our symmetric SWCQ designs in the sense of achieving any point on the sum–rate bounds (9) and (18). The proofs of both theorems are given in Appendix C.

**Theorem 3:** Let \((R^n_1, R^n_2)\) be any point on the sum–rate bound \(\partial R_{SWW}^\text{BT}(D_1^n, D_2^n)\) of (9) for the direct MT problem (assume (10) is satisfied). For any \(\epsilon > 0\), there exists a block length \(n\), two symmetric SWCQ encoders \((\mathcal{E}_1, \mathcal{E}_2)\), which separately compress sources \(Y_1^n\) and \(Y_2^n\) at rates \(R_1^n\) and \(R_2^n\), respectively, and a symmetric SWCQ decoder \(D\), which jointly reconstructs the sources as \(\hat{Y}_1^n\) and \(\hat{Y}_2^n\), such that

\[
\frac{1}{n} \sum_{i=1}^{n} E[(\hat{Y}_i^j - \hat{Y}_i^j)^2] < D_j^n + \epsilon, \quad j = 1, 2
\]

\[
R_j < R^n_j + \epsilon, \quad j = 1, 2.
\]

**Theorem 4:** Let \((R^n_1, R^n_2)\) be any point on the sum–rate bound \(\partial R_{SWW}^\text{BT}(D_1^n)\) of (18) for the indirect MT problem (assume (19) is satisfied). For any \(\epsilon > 0\), there exists a block length \(n\), two symmetric SWCQ encoders \((\mathcal{E}_1, \mathcal{E}_2)\), which separately compress observations \(Y_1^n\) and \(Y_2^n\) at rates \(R_1^n\) and \(R_2^n\), respectively, and a symmetric SWCQ decoder \(D\), which jointly reconstructs source \(X^n\) as \(\hat{X}^n\), such that

\[
\frac{1}{n} \sum_{i=1}^{n} E[(X_i - \hat{X}_i)^2] < D^* + \epsilon
\]

\[
R_j < R^*_j + \epsilon, \quad j = 1, 2.
\]

**IV. PRACTICAL QUANTIZER DESIGN AND HIGH-RATE PERFORMANCE ANALYSIS**

There are two key components in our SWCQ framework: VQ and SW coding. Both of them need to be optimal to achieve the sum–rate bounds in (11) and (20) for the direct and indirect MT problems, respectively; that is, each quantizer must be capable of achieving the rate–distortion limit of its Gaussian input source, and SW coding capable of compressing the two quantized sources to their joint entropy. Additionally, it also requires the two quantization noises to be independent of the sources (and each other) such that the Markov assumptions in the achievability proofs of [25], [26], [38] are satisfied.

It is shown by Zamir and Berger [47] that at high resolution, high-dimensional dithered lattice quantizer [48], [47], [50] can fulfill the above requirements. When the dimensionality goes to infinity, a dithered lattice quantizer can indeed achieve the rate–distortion limit of the Gaussian source, while producing white Gaussian quantization noise that is independent of the source. The use of independently dithered lattice quantizers for direct MT source coding was suggested in [47] so that the quantization noises are mutually independent. However, it is not practical to implement lattice quantizers in high dimension. Fortunately, TCQ [21] provides a suboptimal yet efficient means of realizing high-dimensional VQ. Although TCQ is not strictly a lattice quantizer, it shares many nice properties (e.g., congruent Voronoi regions) with the latter. Another merit of using TCQ is that its dithering sequence can be generated by a simple i.i.d. uniformly distributed source. This reduces the complexity of TCQ when compared to dithered lattice quantization, which requires the dither sequence to be uniformly distributed over the basic Voronoi region. Moreover, except for the trellis bits, the codeword vectors in the TCQ indices are memoryless, making the design of the succeeding SW coder much easier. Therefore, in our practical code design, we use TCQ for all quantizers described in the previous section.

In the remainder of this section, we first review TCQ and show how a dithering sequence can be used in TCQ to produce quantization noise independent of the source, we then perform high-rate performance analysis of our asymmetric and symmetric SWCQ design under practical TCQ and ideal SW coding.

**A. Trellis-Coded Quantization (TCQ)**

TCQ [21] is the source coding counterpart of trellis-coded modulation (TCM) [36]. It is the most powerful practical technique for implementing high-dimensional VQ, due to its excellent MSE performance at modest complexity.

A TCQ is defined by a one-dimensional expanded signal set (ESS) and trellis of a convolutional code. Suppose we want to quantize a continuous source \(X\) using rate \(R\) b/s. TCQ first forms an ESS of size \(2^{2R+1}\) (denoted as \(D\)), and equally partitions it into \(N_c = 2^{2R+1}\) subsets, \(\mathcal{D}\), each having \(2^{R+1}\) points. These \(N_c\) subsets (also referred to as cosets) are denoted as \(D_0, D_1, D_2, \ldots, D_{N_c-1}\), and hence, \(D = \bigcup_{i=0}^{N_c-1} D_i\). In general, the partition of the \(2^{2R+1}\) signal points in \(D\) proceeds from left to right, labeling consecutive points \(D_0, D_1, \ldots, D_{N_c-1}\), \(D_0, D_1, \ldots, D_{N_c-1}\). This way, each signal point in \(D\) can be denoted as \(q_0^c, w = 0, 1, \ldots, 2^{R+1} - 1, c = 0, 1, \ldots, N_c - 1\), where \(c\) is the coset index such that \(q_0^c \in D_c\) and \(w\) the codeword index. A trellis is defined by a possibly time-dependent state transition diagram of a finite-state machine. More precisely, a length-\(n\) rate \(\frac{R}{n}\) trellis \(T\) with \(N^c\) states is a concatenation of \(n\) mappings, where the \(i\)th mapping \((0 \leq i \leq n - 1)\) is from the \(i\)th state of the machine \(S_i\) \((0 \leq S_i \leq N_c - 1)\) and the \(i\)th input \(\tilde{r}_i\)-bit message \(m_i\) to the next state \(S_{i+1}\) and the \(i\)th output \((\tilde{r}_i+1)\)-bit message \(c_i\), i.e., \(T = \{q_0^c\}_{c=0}^{N_c-1}\) with \(q_0^c : (S_i, m_i) \mapsto (S_{i+1}, c_i)\). The trellises used in TCQ are usually time-invariant and are based on an underlying convolutional code \(C\) with rate \(\frac{R}{n+1}\). Under this constraint, we can define a trellis \(T\) by one of its component mappings \(q_0^c \equiv \phi : (S_{\text{current}}^n, m_i) \mapsto (S_{\text{next}}^n, c_i)\), where \(0 \leq m \leq 2^R - 1\) and \(0 \leq c \leq 2^R - 1\). The input–output relation of \(T\) can be written then as \(c = T(m)\).
Based on a size-$2^{R+1}$ ESS $\mathcal{D}$ and a length-$n$ trellis $T$ with $N_s$-state machine, the source $X$ is quantized using the Viterbi algorithm one block $x$ at a time. We associate the $i$th sample $x_i$ in $x$ with the cost $D_{i}(c_i) = \min_{m_i} \| x_i - a_{m_i} \|^2$, which is the distortion between $x_i$ and the codeword in $D_{i}(c_i)$ that is closest to $x_i$. The Viterbi algorithm then searches for the input vector $m = \{m_0, m_1, \ldots, m_{n-1}\}$ that minimizes the accumulated distortion defined as $D(m) = \sum_{i=0}^{n-1} D_{i}(T_{i}(m))$, where $T_{i}(m) = c_i$ is the $i$th trellis output corresponding to the input vector $m$. Finally, TCQ stacks the $R-R$ bit-planes of the codeword vector $w = \{w_0, w_1, \ldots, w_{R-1}\}$ on top of the $R$ bit-planes of trellis vector $m$ to form its output index vector $b = \{b_0, b_1, \ldots, b_{R-1}\}$, achieving a rate of $R/n$ bits, where $b_i = ((b_{R-1}^i, \ldots, b_1^i), (b_{R-2}^i, \ldots, b_0^i))$, with $(b_{R-1}^i, b_1^i)$ and $(b_{R-2}^i, b_2^i)$ coming from the binary representation of $w_i = (b_{R-1}^i, \ldots, b_1^i)$ and $m_i = (b_{R-1}^i, b_2^i)$, respectively. This way, we can denote a trellis-coded quantizer as $b = \mathcal{C}_{D}^{TCQ}(x)$. The above defined TCQ is often referred to as fixed-rate TCQ [21]. Although the ESS of TCQ can be carefully designed according to a specific source distribution, we constrain ourselves to a uniform ESS due to its analytical simplicity and nice properties.

In general, TCQ cannot be classified as a lattice quantizer, because stacking $R+1$ binary linear codes does not necessarily result in a linear code in GF($2^{R+1}$). However, in the special case of $R = 1$ (number of cosets $N_c = 4$), TCQ shares a nice property with the lattice quantizers: congruent Voronoi regions. Indeed, suppose that $\mathcal{C}_{D}^{TCQ}$ is a trellis-coded quantizer with $R = 1$. Then, for any $b, b' \in 2^{R \times n}$, Voronoi region $\mathcal{V}_b = \{x \in \mathcal{X}^n : \mathcal{C}_{D}^{TCQ}(x) = b\}$ is congruent to Voronoi region $\mathcal{V}_{b'} = \{x \in \mathcal{X}^n : \mathcal{C}_{D}^{TCQ}(x) = b'\}$.

Fig. 7(a) is an example of the Voronoi region $\mathcal{V}_b$ of TCQ with $n = 3$, $R = 1$, $N_s = 4$, and $D = \{ -7, -6, \ldots, 0, 1, \ldots, 7, 8 \}$. We can see that $\mathcal{V}_b$ is a nonregular polyhedron with 18 vertices and 12 faces. Fig. 7(b) illustrates how $\mathcal{V}_b$ and its congruent counterparts fill the three-dimensional space. Clearly, the Voronoi regions of TCQ are not simply translations of each other, while those of lattice quantizers are.

In terms of practical performance, TCQ with a trellis of $N_s = 256$ states performs 0.2 dB away from the distortion-rate bound for uniform sources, which is better than any vector quantizer of dimension less than 69 [33]. With the help of entropy coding, the same 0.2-dB gap can be obtained at all rates by entropy-constrained TCQ [20], [33] for any smooth pdf by using carefully designed codebooks. This small performance gap can be further reduce by increasing $N_s$ or $R$, which leads to higher complexity. For example, another 0.1-dB granular gain can be obtained by increasing $N_s$ to $8192$ [43].

### B. Independently Dithered TCQ

TCQ is a powerful and efficient source coding technique; however, there is no guarantee that multiple trellis coded quantizers will produce quantization noises independent of each other (that are also independent of the sources), which is a key requirement in the achievability proofs for the direct and indirect MT source coding [25], [26], [38]. To resolve this issue, we have to consider the possibility of adding a dither to TCQ, just as with the entropy-constraint dithered lattice quantizers. Since TCQ is not a lattice quantizer, classical dithering with uniformly distributed dither over the basic Voronoi region of the lattice no longer produces an independent quantization noise. Thus, we have to find an alternative way of generating a dither sequence of TCQ.

In this subsection, we show that under some mild assumptions, a trellis coded quantizer with an i.i.d. dither sequence can produce independent quantization noise. Without loss of generality, we assume that $R = 1$ and the step size of the ESS is one, i.e., the ESS $D = \{ -2^{R} + 0.5, -2^{R} + 1.5, \ldots, 2^{R} - 0.5 \}$ is partitioned into $N_s = 4$ cosets, each with $2^{R-1}$ points. For a given pdf $f_X(x)$, we define the accumulated distribution of $f_X(x)$ with respect to the ESS $D$ as

$$
\sum_{x \in D_{i}} f_X(x) = \begin{cases} 
   f_X(x - 2^{R} + 4), & x \leq 0 \\
   \sum_{x = -2^{R} + 2}^{x} f_X(x - 4i), & 0 < x \leq 4 \\
   f_X(x + 2^{R} - 8), & x > 4 
\end{cases}
$$

We say that a source distribution $f_X(x)$ is $\Sigma$-uniform with respect to $D$ if $f_X(x)$ is uniformly distributed in the interval $[0, 4]$. Indeed, all symmetric smooth distributions are very close to $\Sigma$-uniform unless the rate $R$ is very low.

The following lemma states that under the $\Sigma$-uniform assumption, a trellis-coded quantizer with an i.i.d. uniform dither sequence in the range of $[-0.5, 0.5]$ can produce independent quantization noises. The proof is given in Appendix D.

**Lemma 1:** Assume $f_{X+V}(x + v)$ is $\Sigma$-uniform with respect to $D$ (with step size 1), where $V$ is a random variable uniformly distributed over $[-0.5, 0.5]$. Define the quantization error as

$$
Q^v = (X + V^n) - [Q_{D}^{TCQ}(X + V^n)]
$$

where $Q_{D}^{TCQ}$ is a trellis-coded quantizer with $R = 1$. Then, as $n$ goes to infinity, $Q_i$ becomes independent of $X_i$ for $0 \leq i \leq n - 1$, i.e.,

$$
\lim_{n \rightarrow \infty} p_{X_i} \cdot Q_i(x_i, q_i) = p_{Q_i}(q_i)
$$

An illustrative comparison between dithered and nondithered trellis-coded quantizers is given in Fig. 8, in terms of the joint
statistics of the $i$th quantization noise $Q_i$ and the $i$th source sample $X_i$. Obviously, dithered TCQ (Fig. 8(a)) produces independent quantization noise, whereas nondithered TCQ (Fig. 8(b)) does not.

Note that for the case with $R > 1$ (i.e., there are more than four costs), Lemma 1 still holds, since a similar symmetry property (as stated in Proposition 1 of Appendix D) exists among the cosets.

C. High-Rate Performance Analysis

Since a practical MT source coding problem is a source-channel coding problem, where quantization is followed by channel coding for SW coding, the total loss contains quantization loss due to source coding and binning loss due to channel coding [41]. In this subsection, we assume ideal binning (via capacity-achieving channel coding), and restrict ourselves to the high-rate/resolution scenario (i.e., $D^*, D^1, D^2 \to 0$). The asymptotical performance of our TCQ-based asymmetric and symmetric SWCQ schemes for both direct and indirect MT source coding can be characterized by the following two theorems. The proofs are given in Appendices E and F, respectively.

**Theorem 5:** Let $(R_1^*, R_2^*)$ be any point on the sum-rate bound $\partial R_{12}^{MT}(D_1^*, D_2^*)$ of (9) for the direct MT source coding problem (assume (10) is satisfied), or $\partial R_{12}^{MT}(D^*)$ of (18) for the indirect MT source coding problem (assume (19) is satisfied), then under ideal SW coding, the achievable rates $R_{21}$, $R_1$, and $R_2$ with our asymmetric SWCQ scheme satisfy

$$R_1 = R_1^* + \frac{1}{2} \log (2\pi e G_{Q_1}) + o(1)$$

$$R_2 = R_{21} + R_{22}$$

$$= R_2^* + \frac{1}{2} \log (2\pi e G_{Q_{21}}) + \frac{1}{2} \log (2\pi e G_{Q_{22}}) + o(1) \quad (40)$$

where $G_{Q_1}$, $G_{Q_{21}}$, and $G_{Q_{22}}$ are the normalized second moments of $Y_1$ for the three employed trellis-coded quantizers $Q_1$, $Q_{21}$, and $Q_{22}$, respectively; and $o(1) \to 0$ as $D^*, D_1^*, D_2^* \to 0$.

**Theorem 6:** Let $(R_1^*, R_2^*)$ be any point on the sum-rate bound $\partial R_{12}^{MT}(D_1^*, D_2^*)$ of (9) for the direct MT source coding problem (assume (10) is satisfied), or $\partial R_{12}^{MT}(D^*)$ of (18) for the indirect MT source coding problem (assume (19) is satisfied), then under ideal SW coding, the achievable sum-rate of our symmetric SWCQ scheme satisfies

$$R_1 + R_2 = R_1^* + R_2^* + \frac{1}{2} \log (2\pi e G_{Q_1}) + \frac{1}{2} \log (2\pi e G_{Q_2}) + o(1) \quad (41)$$

where $G_{Q_1}$ and $G_{Q_2}$ are the normalized second moments of $Y_1$ for the two trellis coded quantizers $Q_1$ and $Q_2$, respectively; and $o(1) \to 0$ as $D^*, D_1^*, D_2^* \to 0$.

Before presenting our practical asymmetric and symmetric SW designs, we point out that our results in Theorems 5 and 6 are consistent with those obtained by Zamir and Berger [47] in their theoretical work on MT source coding at high resolution.

V. PRACTICALASYMMETRIC AND SYMMETRIC SW CODE DESIGNS

The main elements of our practical asymmetric/symmetric SWCQ schemes are dithered TCQ (described in Section IV-B) and asymmetric/symmetric SW coding based on LDPC and turbo codes. We give details of the latter next.

A. Asymmetric SW Code Design

The SW theorem [32] was proved using random binning arguments [9]. The main idea is to randomly partition all length-$n$ sequences into disjoint bins, transmit the index of the bin containing the source sequence, and pick from the specified bin a

---

Fig. 8. Joint statistics of quantization noise $Q_i$ and $X_i$ for TCQ (a) with dither and (b) without dither.
source sequence that is jointly typical with the side information sequence at the decoder. Asymptotically, no rate loss is incurred in SW coding due to the absence of side information at the encoder.

However, there is no efficient decoding algorithm for such a random binning scheme. The first step toward a constructive SW code was given in [39], where the use of a linear parity-check channel code was suggested for partitioning all the source sequences into bins indexed by *syndromes* of a channel code. The set of all valid codewords (with zero syndrome) of the channel code forms only one bin, while other bins are shifts of this zero-syndrome bin. This approach is detailed below.

Let $C$ be an $(n, k)$ binary linear block code with generator matrix $G_{k \times n}$ and parity-check matrix $H_{(n-k) \times n}$ such that $GH^T = 0$. The syndrome of any length-$n$ binary sequence $x$ with respect to code $C$ is defined as $s = xH^T$, which is a length-$(n-k)$ binary sequence. Hence, there are $2^{n-k}$ distinct syndromes, each indexing $2^k$ length-$n$ binary source sequences. A coset $C_s$ of code $C$ is defined as the set of all length-$n$ sequences with syndrome $s$, i.e., $C_s = \{x \in \{0,1\}^n : xH^T = s\}$.

Consider the problem of SW coding of a binary source $X$ with decoder side information $Y$ (with discrete [32] or continuous [17] alphabet). Syndrome-based SW coding of $x$ proceeds as follows.

- **Encoding**: The encoder computes the syndrome $s = xH^T$ and sends it to the decoder at rate $R_{SW} = \frac{n-k}{n}$ b/s. By the SW theorem [32]
  \[ R_{SW} = \frac{n-k}{n} \geq H(X|Y), \quad (42) \]

- **Decoding**: Based on the side information $y$ and received syndrome $s$, the decoder finds the most probable source sequence $\hat{x}$ in the coset $C_s$, i.e.,
  \[ \hat{x} = \arg \max_{x \in C_s} P(x|y). \quad (43) \]

This syndrome-based approach was first implemented by Pradhan and Ramchandran [27] using block and trellis codes. More advanced channel codes, such as turbo codes, are later used for asymmetric SW coding [1], [2], [19] to achieve better performance. Following the work in [18], we consider using LDPC codes [12], not only because of their capacity-approaching performance, but also their flexible code designs using density evolution [29]. Another reason for our choice lies in low-complexity decoding based on the message-passing algorithm, which can be applied in SW coding with only slight modification [18]. Specifically, as in the conventional message-passing algorithm, the input for the $i$th variable node is the log-likelihood ratio (LLR) of $x_i$ defined as
  \[ \Lambda_{ch}(x_i) = \log \frac{P(X = 0 | Y = y_i)}{P(X = 1 | Y = y_i)}, \quad 0 \leq i \leq n - 1. \quad (44) \]

If $X$ is uniform with $P(X = 1) = P(X = 0) = 1/2$, we have
  \[ \Lambda_{ch}(x_i) = \log \frac{P(Y = y_i | X = 0)}{P(Y = y_i | X = 1)}, \quad 0 \leq i \leq n - 1. \quad (45) \]

The $j$th syndrome bit $s_j$, $0 \leq j \leq n - k - 1$, is in fact the binary sum of the source bits corresponding to the ones in the $j$th row of the parity-check matrix $H$. Hence, the $j$th check node in the Tanner graph is related to $s_j$. The only difference from conventional LDPC decoding is that one needs to flip the sign of the check-to-bit LLR if the corresponding syndrome bit $s_j$ is one [18]. Moreover, conventional density evolution [29] can be employed to analyze the iterative decoding procedure without any modification [7].

### B. Symmetric SW Code Design

Our symmetric SWCQ design consists of dithered TCQ followed by symmetric SW coding (hence the name symmetric SWCQ) based on turbo/LDPC codes. In the remaining part of this section, we describe the employed symmetric SW coding scheme based on the channel partitioning method of [31], elaborate on our novel multilevel symmetric SW coding framework for compressing different bit-planes of quantization indices, and compute the loss of the SWCQ design due to practical coding.

#### 1) Symmetric SW Coding for Uniform Binary Sources [31]:

Let $J$ and $K$ be two memoryless uniform binary sources. They are related by a binary-symmetric channel (BSC) with crossover probability $P(J \oplus K = 1) = p$, where $\oplus$ denotes binary addition. Our goal is to separately compress $J$ and $K$, and to jointly reconstruct them. Due to the SW theorem [32], any rate pair $(r_1, r_2)$ that satisfies
  \[ r_1 \geq H(J|K) = H(p) \]
  \[ r_2 \geq H(K|J) = H(p) \]
  \[ r_1 + r_2 \geq H(J,K) = H(p) + 1 \quad (46) \]

is achievable. In [31], an efficient algorithm to design good symmetric SW codes by partitioning a single linear parity-check code was proposed. Although this algorithm can be applied to compression of multiple correlated sources, we restrict ourselves to two sources only.

Suppose that we aim at approaching a point $(r_1, r_2)$ (i.e., to compress $J$ at rate $r_1$ and $K$ at $r_2$) that satisfies (46). Let $C$ be an $(n, k)$ linear channel block code with $k = (2 - r_1 - r_2)n$. Although both systematic and nonsystematic codes can be used for $C$ [31], for the sake of easy implementation, we assume that $C$ is a systematic channel code with generator matrix $G = [I_k, P_{k \times (n-k)}]$. We partition $C$ into two subcodes, $C_1$ and $C_2$, defined by generator matrices
  \[ G_1 = [I_{m_1}, O_{m_1 \times m_2} P_1] \quad \text{and} \quad G_2 = [O_{m_2 \times m_1} I_{m_2} P_2] \]

which consist of the top $m_1$ and bottom $m_2$ rows of $G$, respectively, where $m_1 \triangleq 1 - r_1 n$, $m_2 \triangleq 1 - r_2 n$ (thus, $m_1 + m_2 = k$). Then the parity-check matrices for $C_1$ and $C_2$ can be written as
  \[ H_1 = \begin{bmatrix} O_{m_2 \times m_1} & I_{m_2} & O_{m_2 \times (n-k)} & I_{n-k} \end{bmatrix} \]
  \[ H_2 = \begin{bmatrix} I_{m_1} & O_{m_1 \times m_2} & O_{m_1 \times (n-k)} & I_{n-k} \end{bmatrix} \]

respectively.

2) Encoding: It is done by multiplying $U^n$ and $V^n$, the realization of $J^n$ and $K^n$, respectively, by the corresponding parity-check matrix $H_1$ and $H_2$, respectively. We partition
the length-\(n\) vectors \(u^n\) and \(v^n\) into three parts (which are of lengths \(m_1, m_2\), and \(n - k\))

\[
u^n = \begin{bmatrix} u_1^{m_1} & u_2^{m_2} & u_3^{n-k} \end{bmatrix}, \quad v^n = \begin{bmatrix} v_1^{m_1} & v_2^{m_2} & v_3^{n-k} \end{bmatrix}, \quad (49)
\]

Then, the resulting syndrome vectors are

\[
s_1^{n-m_1} = u^n H_T^T = \begin{bmatrix} u_3^{n-k} & u_1^{m_1} P_1 \end{bmatrix}
\]
\[s_2^{n-m_2} = v^n H_T^T = \begin{bmatrix} v_3^{m_2} & v_1^{m_1} P_2 \end{bmatrix}
\]

which are directly sent to the decoder. It is easy to see that the total number of transmitted bits for \(u^n\) and \(v^n\) is \(m_2 + (n - k) = nr_1\) and \(m_1 + (n - k) = nr_2\), respectively, with the desirable sum–rate of \(r_1 + r_2\) b/s.

3) Decoding: Upon receiving the syndrome vectors \(s_1^{n-m_1}\) and \(s_2^{n-m_2}\), the decoder forms an auxiliary length-\(n\) row vector as

\[
s^n = \begin{bmatrix} v_1^{m_1} & u_2^{m_2} & (u_3^{n-k} + v_3^{n-k}) & u_1^{m_1} P_1 & v_2^{m_2} P_2 \end{bmatrix}
\]

\[= \begin{bmatrix} u_1^{m_1} & u_2^{m_2} & (u_3^{n-k} + v_3^{n-k}) & [u_1^{m_1} v_2^{m_2}] P \end{bmatrix}
\]

Then it finds a codeword \(c^n\) of the main code \(C\) closest (in Hamming distance) to \(s^n\). Let the vector \([\theta_1^{m_1} \theta_2^{m_2}]\) be the systematic part of \(c^n\), then \(u^n\) and \(v^n\) are recovered as

\[
\theta^n = \begin{bmatrix} \theta_1^{m_1} G_1 \oplus O_{1 \times m_1} & \theta_2^{m_2} G_2 \oplus O_{1 \times m_2} \end{bmatrix}
\]

\[= \begin{bmatrix} \theta_1^{m_1} G_1 & \theta_2^{m_2} G_2 \end{bmatrix}
\]
\[= \begin{bmatrix} \theta_1^{m_1} & \theta_2^{m_2} \end{bmatrix}
\]

\[= \begin{bmatrix} \theta_1^{m_1} & \theta_2^{m_2} \end{bmatrix}
\]

(51)

It is shown in \([31]\) that if the \((n, k)\) main code \(C\) approaches the capacity of a BSC with crossover probability \(p\), then the above designed symmetric SW code approaches the SW limit for the same binary-symmetric correlation channel model.

4) Correlation Model Between \(B_1\) and \(B_2\): To apply the above symmetric SW coding scheme, first, we have to model the correlation between outputs of the two dithered quantizers \(B_1^n\) and \(B_2^n\). Clearly, this correlation is uniquely determined by the pair of dither sequences \((V_1^n, V_2^n)\) used in the two quantizers. Now fix a pair of \((V_1^n, V_2^n)\), and expand the trellis bit-plane \((J_1^n, K_1^n)\) to the corresponding coset index sequences \((C_1^n, C_2^n) = \begin{bmatrix} Q_1(n) & 0 \end{bmatrix} C_1^n \begin{bmatrix} 0 & Q_2(n) \end{bmatrix} = \begin{bmatrix} Q_1(n) & 0 \end{bmatrix} C_2^n \begin{bmatrix} 0 & Q_2(n) \end{bmatrix})\), then correlation modeling is done on the sample level by computing the joint probability mass function (pdf) \(P(B_1^n, B_2^n)\), where \(B_1^n\) and \(B_2^n\) are the indices of the signal points \(q_{C_1^n}^{W_{C_1}}\) and \(q_{C_2^n}^{W_{C_2}}\) (the ESS \(\mathcal{D}\) are the same for both quantizers), respectively, to which the sources are quantized, namely

\[
B_1 = W_{Q_1} 4 + C_{Q_1} = (J_1(J_1-1) \ldots J_2) 4 + C_{Q_1},
\]

\[
B_2 = W_{Q_2} 4 + C_{Q_2} = (K_2 K_2-1) \ldots J_2) 4 + C_{Q_2}, \quad (52)
\]

One possible solution to compute \(P(B_1, B_2)\) is to collect empirical statistics of \((B_1, B_2)\) by counting the number of occurrences of each quantization index pair \((B_1, B_2)\) based on the quantization output of training data generated according to the joint pdf of \((Y_1, Y_2)\). This method is similar to that used in \([43]\). However, to get a good approximate of two-dimensional pmf \(P(B_1, B_2)\) using empirical statistics, we need a large number of Monte Carlo simulations, which is a time consuming, especially when the number of quantization levels is large, which is the case in the high-rate regime we consider.

A simpler solution can be obtained by assuming a Markov chain \(B_1 \rightarrow Y_1 \rightarrow Y_2 \rightarrow B_2\), where \(Y_1 = Y_1 + V_1\) and \(Y_2 = Y_2 + V_2\) are the actual inputs to the two TCQ quantizers. That is \(P_{B_1|Y_1}(B_1|Y_1)\) and \(P_{B_2|Y_2}(B_2|Y_2)\) are the one-dimensional output–input relationship of nondithered TCQ, which can be approximated using the method described in \([43]\). Specifically, we write \(P_{B_1|Y_1}(B_1|Y_1)\) as \((54)\)–\((55)\), shown at the bottom of the page, where the real line \(\mathbb{R}\) is partitioned into \(2T + 1\) length-\(\theta\) intervals (except two boundary ones): \(\Theta_{-T} \cup \Theta_{-T+1} \cup \ldots \cup \Theta_T\), with \(\Theta_i, i = -T, \ldots, T\), being the middle point of the \(i\)th interval \(\Theta_i\). Note that the last approximation in \((55)\) may be inaccurate if \(\theta\) is not small enough or the correlation coefficient \(\rho\) is very close to 1. Under these circumstances, we can resort to the numerical method described in \([10]\) to compute the bivariate Gaussian probability

\[
\int_{\Theta_i} \int_{\Theta_j} p_{T_1, T_2}(\bar{Y}_1, \bar{Y}_2) \ d\bar{Y}_1 d\bar{Y}_2.
\]

An example of the resulting joint pmf \(p(B_1, B_2)\) computed using \((54)\) with \(V_1 = V_2 = 0\) and the number of bit planes \(m = 3\) is plotted in Fig. 9. Note that, because of the symmetry assumptions on the sources (recall that we assume \(\sigma^2_{Y_1} = \sigma^2_{Y_2}\) in the direct case and \(\sigma^2_{R_1} = \sigma^2_{R_2}\) in the indirect case) and the quantizers (the same quantization step size \(q\)), \(P(B_1, B_2)\) is symmetric with respect to the diagonal line on the \((B_1, B_2)\) plane. We also observe that most of the probability mass is concentrated near the
diagonal line, because the quantization outputs of the two correlated sources/noisy observations, \( Y_1 \) and \( Y_2 \), are still correlated. Based on \( p(B_1, B_2) \), we develop a multilevel coding framework for SW coding of the bit-planes of \( B_1 \) and \( B_2 \).

5) Multilevel Symmetric SW Coding Framework: Let \( (J_1, \ldots, J_m) \) and \( (K_1, \ldots, K_m) \) be binary representations of \( B_1 \) and \( B_2 \), respectively. \( J_1 \) and \( K_1 \) are the trellis bit-planes, used to specify one of the four cosets for each sample. The rest are codeword bit-planes, which are the output of the scalar quantizer with the specified coset as its codebook. Hence, given a trellis bit-plane, all codeword bit-planes are memoryless.

Then, from the chain rule, we have

\[
H(B_1, B_2) = H(J_1, \ldots, J_m, K_1, \ldots, K_m)
= H(J_1, K_1) + \sum_{j=2}^{m} H(J_j, K_j | M_{j-1})
\]

where \( M_{j-1} = (J_{j-1}, K_{j-1}, \ldots, J_1, K_1) \). To introduce flexibility in the rate allocation between the two encoders, we employ the symmetric SW code design based on channel code partitioning [31] for each bit-plane of \( B_1 \) and \( B_2 \). Note that if we assume ideal source coding (with independent dithering) and ideal SW coding, \( Y_1, Y_2, \tilde{Z}_1, \tilde{Z}_2 \) are jointly Gaussian. In this case, \( H(B_1, B_2) = I(Y_1; Y_2; \tilde{Z}_1; \tilde{Z}_2) \) is the sum-rate bound defined in (9) and (18); hence, we have the following lemma. The proof is straightforward (hence omitted) by considering two extreme cases of multilevel symmetric SW coding when we attempt to allocate the minimum rate to \( B_1 \) or \( B_2 \).

**Lemma 2:** For fixed dithered quantizers \( Q_1 \) and \( Q_2 \) with outputs \( B_1 = \{J_1, \ldots, J_m\} \) and \( B_2 = \{K_1, \ldots, K_m\} \), any rate pair \( (R_1, R_2) \) that satisfies \( R_{\text{min}} \leq R_1, R_2 \leq R_{\text{max}} \) where

\[
R_{\text{max}}^e = H(J_1) + \sum_{j=2}^{m} H(J_j | M_{j-1}),
\]

\[
R_{\text{min}}^e = H(K_1 | J_1) + \sum_{j=2}^{m} H(K_j | M_{j-1}, J_j)
\]

is potentially achievable with our multilevel symmetric SW codes.

If we compute the difference between \( (R_{\text{max}}^e, R_{\text{min}}^e) \) and one of the corner points on the inner sum-rate bound, which is \( (R_{\text{max}}^e, R_{\text{min}}^e) = (H(B_1), H(B_2|B_1)) \), we have a gap of

\[
\Delta_R = R_{\text{max}}^e - R_{\text{tr}_{\text{max}}}^e = R_{\text{min}}^e - R_{\text{tr}_{\text{min}}}^e
= \sum_{j=2}^{m} I(J_j; K_{j-1} \ldots K_1 | J_{j-1} \ldots J_1) \geq 0.
\]

This gap comes from the different coding order between multilevel symmetric and asymmetric SW coding in the extreme cases. Our experiments show that this gap is very small in practice (e.g., 0.03 b/s). One possible improvement of this pure symmetric design is to use asymmetric SW coding for some of the bit-planes. If we carefully design the order of SW coding, the resulting SWCQ design not only can approach more points on the inner sum-rate bounds than the symmetric SWCQ design, but also has better practical performance.

6) Practical Implementation: In practice, there is a rate loss due to the suboptimality of TCQ. In addition, compressing trellis bit-planes \( J_1, K_1 \) to \( H(J_1, K_1) \) b/s is very difficult because of the lack of a mechanism for exploiting the memory in these trellis bits in practical SW coding. We thus send \( J_1 \) and \( K_1 \) to the decoder using one b/s for each and incur some loss in rate (note that for the two-bit variables \( J_1 \) and \( K_1 \), the second bit is a function of the first bit).

For SW coding of \( J_j \) and \( K_j, 2 \leq j \leq m \), the symmetric SW code design in [31] cannot be directly applied because the correlation between \( J_j \) and \( K_j \) conditioned on \( M_{j-1} \) is more complex than the BSC correlation model exploited in [31]. Our proposed multilevel coding framework generalizes the approach of [31] in terms of handling more general correlation models, while still enjoying the desirable property of arbitrarily allocating the total number of output syndrome bits between the two encoders. The key novelties lie in the construction of
TABLE I

<table>
<thead>
<tr>
<th>$J_1 \setminus K_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9959</td>
<td>0.9958</td>
<td>0.5019</td>
<td>0.0451</td>
</tr>
<tr>
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<td>0.9958</td>
<td>0.9553</td>
<td>0.5045</td>
</tr>
<tr>
<td>2</td>
<td>0.5028</td>
<td>0.9561</td>
<td>0.9958</td>
<td>0.9567</td>
</tr>
<tr>
<td>3</td>
<td>0.0456</td>
<td>0.5037</td>
<td>0.9562</td>
<td>0.9959</td>
</tr>
</tbody>
</table>

lookup tables for the probabilities $\{P(J_1 = 1|K_1 = 0, M_{j-1})$, $P(J_1 = 0|K_1 = 1, M_{j-1})$, $P(K_1 = 0|J_1 = 1, M_{j-1})$, $P(K_1 = 0|J_1 = 0, M_{j-1})$, $P(J_1 = 0|K_1 = 0, M_{j-1})\}$, which are used for computing the LLRs at the multilevel channel decoder. An example of the lookup table for $P(J_2 = 0|M_1)$ (recall that $M_1 = \{J_1, K_1\}$) is given in Table I.

According to [31], part of the SW-coded syndrome bits for $J_j$ and $K_j$ consists of a portion of the uncompressed $J_j$ and $K_j$ (see (50)). To exploit cross-bit-plane correlation among the codeword bits $\{J_2, \ldots, J_n\}$ (and likewise among codeword bits $\{K_2, \ldots, K_n\}$), we employ adaptive arithmetic coding separately at each encoder to compress this part of the syndrome bits from 1 b/s to $H(J_n|J_1, \ldots, J_j)$ (or $H(K_n|K_1, \ldots, K_j)$) b/s. The remaining syndrome bits are sent to the decoder without further compression. Note that the $j$th bit-plane $J_j$ (or $K_j$), $2 \leq j \leq m$, is compressed with rate $r_j$ (or $r_j$) using the symmetric SW coding scheme of [31] outlined in Section V-B.1 while assuming all previously reconstructed bit-planes as decoder side information. Thus, we design an $(n, k_j)$ linear block code $C_j$ with $k_j = \frac{2^{n-j} - 2}{2^{n-j} - 2} n$, where $h_j \triangleq \frac{1}{n} H(J^n; J_1^{n-1}, \cdots, J_j)$, $h_j \triangleq \frac{1}{n} H(K^n; K_1^{n-1}, \cdots, K_j)$: we set $r_j > r_j$ to ensure $k_j < n$.

VI. SIMULATION RESULTS

A. Asymmetric SWCQ

For the direct MT source coding problem, sources $X_1$ and $X_2$ are zero mean, jointly Gaussian with variances $\sigma_{21}^2 = 1$ and correlation coefficient $\rho = 0.99$. The target distortions $D_1$ and $D_2$ are both set to be 0.01, then the sum-rate bound $\partial D_{12}^B (D_1, D_2)$ for the direct MT problem can be computed using (9) as

$$R_1 + R_2 \geq \frac{1}{2} \log^+ \left( \frac{1 - \rho^2}{2} \frac{\sigma_{21}^2 \sigma_{22}^2}{\sigma_{21}^2} \right) = 7.142 \text{ b/s}. \quad (59)$$

Suppose that we are attempting to approach the middle point of the sum-rate bound $\partial D_{12}^B (D_1, D_2)$, i.e., $R_1 \approx R_2 \approx 7.142/2 = 3.571 \text{ b/s}$. Then using (69)–(71) and (74), we can compute the three quantization distortions $(d_1, d_1, d_2)$ (assuming ideal quantization) and the minimum MSE coefficients $(\alpha_1, \beta_1, \gamma_1)$. $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3)$, yielding

$$d_1 = 0.14937908, \quad d_1 = 0.00105018, \quad d_2 = 0.00105762; \quad \alpha_1 = 0.05230983, \quad \beta_1 = 0.00032065, \quad \gamma_1 = 0.04674618; \quad \alpha_2 = 0.04707583, \quad \beta_2 = 0.00066912, \quad \gamma_2 = 0.94572981; \quad \alpha_3 = 0.86743343, \quad \beta_3 = 0.12288739. \quad (60)$$

For the indirect MT source coding problem, source $X$ and noises $N_1$ and $N_2$ are zero mean, jointly Gaussian, and mutually independent with variances $\sigma_x^2 = 1$, $\sigma_{21}^2 = \sigma_{21}^2 = 1$ and $\sigma_{22}^2 = \sigma_{22}^2 = 1$ respectively. Noisy observations are given by $Y_1 = X + N_1$ and $Y_2 = X + N_2$. We refer to the ratio $\sigma_{2}^2/\sigma_{22}^2 = 99 = 19.96 \text{ dB}$ as correlation signal-to-noise ratio (CSNR). The target distortion is set to $D^* = 0.00555 = 22.58 \text{ dB}$.

The sum-rate bound $\partial D_{12}^B (D^*)$ for the indirect MT problem can be computed using (18) as

$$R_1 + R_2 \geq \frac{1}{2} \log^+ \left[ \frac{\sigma_{21}^2 \sigma_{22}^2}{\sigma_{22}^2} \left( \frac{4 \rho^2}{\sigma_{22}^2} - \frac{1}{\sigma_{21}^2} + \frac{1}{\sigma_{22}^2} \right) \right] = 7.142 \text{ b/s}. \quad (61)$$

Due to (77)–(80), one can verify that $(d_1, d_1, d_2)$ are scaled versions of those in (60) with scaling factor $\rho^2/\sigma_{22}^2 = 99$, and $(\alpha_1, \beta_1, \gamma_1)$ are the same as those in (60), while $(\alpha_2, \beta_2, \gamma_2)$ are computed using (80) as

$$\alpha_2 = 0.49727666, \quad \beta_2 = 0.00349566, \quad \gamma_2 = 0.49372983. \quad (62)$$

In our implementation, to get the quantization distortions $(d_1, d_1, d_2)$ in (60), we employ three dithered TCQ quantizers with parameters

1. $Q_1: \hat{R}_{TCQ} = 5 \text{ b/s}, \text{ step size } \Delta_1 = 0.7859$
2. $Q_2: \hat{R}_{TCQ} = 7 \text{ b/s}, \text{ step size } \Delta_2 = 0.0657$
3. $Q_3: \hat{R}_{TCQ} = 7 \text{ b/s}, \text{ step size } \Delta_3 = 0.053$

The transmission rates with ideal SW coding, i.e., $R_1 = \frac{1}{2} H(B_{12})$, $R_1 = \frac{1}{2} H(B_{12} \mid V_{12})$, and $R_2 = \frac{1}{2} H(B_{22} \mid V_{12})$ are computed using Monte Carlo simulations. Practical SW encoders are based on turbo and irregular LDPC codes, which are designed such that the decoding bit-error rate is less than $10^{-4}$. In our simulations, the block length (BL) for both turbo and LDPC codes equals to $10^6$, and the maximum number of iterations is set to 100 for turbo decoding and 500 for LDPC decoding. Table II shows the resulting bit-plane-level conditional entropies and the practical SW coding rates. With turbo-based asymmetric SW coding, the total transmission rate $R_1 + R_1 + R_2 = 1,506 + 3,650 + 2,180 = 7,336 \text{ b/s}$. Practical distortions are $(D_1, D_2) = (-30.05 \text{ dB}, -30.01 \text{ dB})$ for the direct setting and $D = -22.60 \text{ dB}$ for the indirect setting, satisfying the target distortion constraints. Hence, our asymmetric SWCQ design based on turbo codes performs $7.336 - 7.142 = 0.194 \text{ b/s}$ away from both sum-rate bounds $\partial D_{12}^B (D_1, D_2)$ for the direct setting and $\partial D_{12}^B (D^*)$ for the indirect setting. With LDPC-based asymmetric SW coding, the total transmission rate is $R_1 + R_1 + R_2 = 1,506 + 3,625 + 2,152 = 7,281 \text{ b/s}$, which is $7.281 - 7.142 = 0.139 \text{ b/s}$ away from both sum-rate bounds. These results together with the sum–rate bounds for both the direct and indirect MT settings are depicted in Fig. 11.

The loss of 0.139 b/s for the best results with LDPC-based asymmetric SW coding consists of three 0.03-b/s losses (corresponding to the 1.34-dB granular gain of 256-state TCQ, or roughly 0.19-dB loss in distortion) from the suboptimality of TCQ, a total of 0.04-b/s loss (see Table II) from practical arithmetic/SW coding, and a very small loss from the jointly Gaussian assumption of the two quantized versions at the two encoders.
TABLE II
ENTROPIES VERSUS PRACTICAL RATES AT HIGH RATE FOR DIRECT AND INDIRECT MT CODING USING ASYMMETRIC SWCQ

<table>
<thead>
<tr>
<th>Quantizer</th>
<th>Bit Plane #</th>
<th>Entropy Turbo (BL = 10^8)</th>
<th>Practical Rate (b/s)</th>
<th>Irregular LDPC Code Profile</th>
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</thead>
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<td></td>
<td>All</td>
<td>1.504</td>
<td>1.506</td>
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<td>3.650</td>
</tr>
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<td>Q_22</td>
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<td>1.000</td>
</tr>
<tr>
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<td></td>
<td>2</td>
<td>0.908</td>
<td>0.925</td>
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<td></td>
<td></td>
<td>3</td>
<td>0.223</td>
<td>0.250</td>
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<td></td>
<td></td>
<td>4</td>
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<td>0.005</td>
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<tr>
<td></td>
<td></td>
<td>5</td>
<td>0.0000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>0.0000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>0.0000</td>
<td>0.000</td>
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<tr>
<td></td>
<td></td>
<td>All</td>
<td>2.131</td>
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<tr>
<td>Total</td>
<td></td>
<td></td>
<td>-</td>
<td>7.241</td>
</tr>
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</table>

B. Symmetric SWCQ

In the implementation of symmetric SWCQ scheme, we use the same set of source distributions and target distortions as in Section VI-A, namely

Direct setting: \(\sigma_{g1}^2 = \sigma_{g2}^2 = 1, \rho = 0.9, D_1^4 = D_2^4 = 0.001\)

Indirect setting: \(\sigma_{z1}^2 = 1, \sigma_{z2}^2 = \sigma_{n2}^2 = \frac{1}{99}, D_1^{\text{IR}} = 0.00555.; \) (63)

Then the sum–rate bounds \(\partial \mathcal{R}_{\text{SWCQ}}^{\text{IR}}(D_1^{\text{IR}}, D_2^{\text{IR}})\) for the direct setting and \(\partial \mathcal{R}_{\text{SWCQ}}(D^*)\) for the indirect setting are both \(R_1 + R_2 \geq 7.142\text{b/s},\) and the two quantization distortions are \(d_1 = d_2 = 0.0010502.\) We employ two identical dithered TCQ quantizers with parameters

1) \(Q_1 : R_{\text{TCQ}} = 7 \text{ b/s}, \) step size \(\Delta_1 = 0.06570;\)

2) \(Q_2 : R_{\text{TCQ}} = 7 \text{ b/s}, \) step size \(\Delta_2 = 0.06570.\)

The conditional entropies for the seven bit-planes of \(B_1\) and \(B_2\) are shown in Table III (due to the symmetry between the sources and encoders, \(J_1\)'s and \(K_2\)'s are interchangeable).

In our practical SW code implementation based on turbo codes, the code length \(n\) equals \(10^6\), and we control the transmission rates such that the decoding probability of error is less than \(10^{-6}\) after 100 iterations. In the bottom-up order, the seven bit planes of \(B_1\) and \(B_2\) are coded in the following way.

1) The first bit plane \((J_1, K_1)\) is directly transmitted using 2 b/s.
The total rate loss due to practical SW coding is 0.035 b/s, 0.64 b/s, and 0.995 b/s, respectively. Since the LDPC code we consider.

The subtotal in rate loss due to practical SW coding is 0.035 b/s, 0.64 b/s, and 0.995 b/s, respectively. Since turbo code rates of length 10^5 bits perform slightly worse than that based on turbo codes (of length 10^6 bits), with a SW rate loss of 0.065 b/s compared to 0.060 b/s, this is due to the shorter block length with LDPC codes. Indeed, at the same block length of 10^5, LDPC code based scheme performs 0.000 b/s – 0.065 = 0.025 b/s better than the turbo based scheme, as shown in Table III.

### C. Low-Rate Performance and Complexity Analysis

We next evaluate the performance of our asymmetric and symmetric SWCQ schemes at low transmission rate, and compare the results to those in [28] for the indirect MT problem at a practical sum-rate of 4.0 b/s.

In our simulations for symmetric SWCQ, CSNR is set to $\sigma^2_n/\sigma^2_n = 90 = 19.96$ dB, and the target distortion is $D^* = -18.58$ dB. Then the sum-rate bound $\partial R^*_1(D^*) = 3.728$ b/s. Practical results with LDPC code based symmetric SWCQ coding are shown in Table IV, where block length $n = 10^5$. The total transmission rate is 3.999 b/s, which is 0.27 b/s away from the sum-rate bound $\partial R^*_1(D^*)$. At the same sum-rate and CSNR, the scheme in [28] can achieve distortion of $-16.3$ dB, which corresponds to a theoretical sum-rate of 3.048 b/s, and is more than 2 dB worse than our results.

In our simulations for asymmetric SWCQ, CSNR is set to $\sigma^2_n/\sigma^2_n = 90 = 19.96$ dB, and the target distortion is $D^* = -18.30$ dB. Then the sum-rate bound $\partial R^*_1(D^*) = 3.728$ b/s. Practical results with LDPC code based asymmetric SWCQ coding are shown in Table V, where block length $n = 10^5$. The total transmission rate is 3.999 b/s, which is 0.27 b/s away from the sum-rate bound $\partial R^*_1(D^*)$. At the same sum-rate and CSNR, the scheme in [28] can achieve distortion of $-16.3$ dB, which corresponds to a theoretical sum-rate of 3.048 b/s, and is more than 2 dB worse than our results.
TABLE V

<table>
<thead>
<tr>
<th>Quantizer</th>
<th>Bit Plane #</th>
<th>Conditional Entropy (b/s)</th>
<th>Practical Rate (b/s)</th>
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<tr>
<td>Q&lt;sub&gt;1&lt;/sub&gt;</td>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
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<tr>
<td></td>
<td>2</td>
<td>0.822</td>
<td>0.840</td>
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<td></td>
<td>3</td>
<td>0.077</td>
<td>0.090</td>
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<tr>
<td></td>
<td>4</td>
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<td>0.000</td>
</tr>
<tr>
<td>All</td>
<td></td>
<td>1.799</td>
<td>1.830</td>
</tr>
<tr>
<td>Q&lt;sub&gt;2&lt;/sub&gt;</td>
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<td>0.861</td>
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<tr>
<td></td>
<td>2</td>
<td>0.053</td>
<td>0.060</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.000</td>
<td>0.000</td>
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<tr>
<td></td>
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<tr>
<td>All</td>
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</tr>
<tr>
<td>Total</td>
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<td>3.934</td>
<td>4.001</td>
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TABLE VI

<table>
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<tr>
<th>SWCQ scheme</th>
<th>Encoding</th>
<th>Decoding</th>
</tr>
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<tr>
<td></td>
<td>Time (s/cond)</td>
<td>Memory (MByte)</td>
</tr>
<tr>
<td>Asymmetric (BL=10^6)</td>
<td>366</td>
<td>75.1</td>
</tr>
<tr>
<td></td>
<td>365</td>
<td>21.5</td>
</tr>
<tr>
<td>Symmetric (BL=10^5)</td>
<td>25.3</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>25.0</td>
<td>2.1</td>
</tr>
</tbody>
</table>

inefficiency in compressing the trellis bit planes using asymmetric SW coding [43]. However, the overall distortion \( D^* = -18.30 \text{ dB} \) with asymmetric SWCQ is still much better than the \(-16.3 \text{ dB} \) performance in [28] at the same sum-rate and CSNR.

Complexity-wise, the best results of [28] for a sum-rate of 4 b/s are obtained with eight-level Lloyd–Max fixed-length scalar quantizer and 32-state trellis codes, while our asymmetric SWCQ scheme employs 256-state TCQ and irregular LDPC codes. Quantitatively, the running time on an Intel Core 2 Duo 1.8-GHz machine and peak memory usage are shown in Table VI.

VII. CONCLUSION

Extending our previous results on practical SW coding [18], [19], [31], and WZ coding [43], we have developed a general SWCQ framework for MT source coding and detailed practical code designs. Assuming ideal source coding (with independent dithering) and ideal SW coding, we have shown that our asymmetric design can achieve any point on the sum–rate bound of the rate regions for both the quadratic Gaussian direct and indirect MT source coding problems, while the symmetric design can approach most of the points. We have also provided an improved SWCQ design that can approach more points and has better performance. Our practical results are very close to the theoretical limits.

Compared to asymmetric SWCQ that involves source splitting, symmetric SWCQ is conceptually simpler, because it only has one quantization step and one SW coding step, and more elegant, because all compression is done in one step that includes both classic entropy coding and syndrome-based channel coding for compression. However, our practical results using LDPC codes for the asymmetric scheme (with a 0.139-b/s gap to the sum–rate bound) performs slightly better than the symmetric scheme (with a 0.157-b/s gap to the sum–rate bound), because the asymmetric scheme benefits from the longer block length (10^6 bits) than the symmetric scheme (10^5 bits). Moreover, there are other extra losses in the symmetric SWCQ design, one of which comes from the assumption that (53) holds; another loss stems from the inefficiency of the symmetric SW code designs of [31] for (conditionally) nonuniform sources.

Finally, we point out that TCQ and SWC coding in our proposed SWCQ framework are designed separately. This is proofably optimal at high rate (see Section IV). At low rate, a separate design is not optimal, and improved performance than those reported in Section VI-C can be obtained by exploiting the non-Gaussian statistics of TCQ indices and employing nonlinear estimation at the joint decoder (as done in [43] for WZ coding).

APPENDIX A

PROOF OF THEOREM 1

Before proving Theorem 1, we first state the following lemma.
Lemma 3: Define three jointly Gaussian random variables \((Z_1, Z_{21}, Z_{22})\) as
\[
Z_1 = Y_1 + Q_1, \quad Z_{21} = Y_2 + Q_{21}, \quad Z_{22} = Y_2 + Q_{22}
\]
where \(Q_1, Q_{21}, Q_{22}\) are zero-mean independent Gaussian random variables that are also independent of \(Y_1\) and \(Y_2\). For any \(\epsilon > 0\), there exists sufficiently large \(n\), asymmetric SWCQ encoders \(E_1, E_2\), and an asymmetric SWCQ decoder \(D\), such that the transmission rates \(R_1\) and \(R_2\) satisfy
\[
R_1 < I(Y_1; Z_1) - I(Z_1; Z_{21}) + \epsilon \quad \text{(65)}
\]
\[
R_2 < I(Y_2; Z_{21}) + I(Y_2; Z_{22}) - I(Z_{22}; Z_1 Z_{21}) + \epsilon \quad \text{(66)}
\]
with average distortions
\[
E\left\{\frac{1}{n} \sum_{i=1}^{n} (Y_{1,i} - \hat{Y}_{1,i})^2\right\} < E\left\{\frac{1}{n} \left( Y_1 - E[Y_1|Z_1, Z_{21}] \right)^2 \right\} + \epsilon \quad \text{(67)}
\]
\[
E\left\{\frac{1}{n} \sum_{i=1}^{n} (Y_{2,i} - \hat{Y}_{2,i})^2\right\} < E\left\{\frac{1}{n} \left( Y_2 - E[Y_2|Z_1, Z_{21}] \right)^2 \right\} + \epsilon. \quad \text{(68)}
\]

Proof: This lemma is a direct consequence of results in [4], [22], [35], [38], hence the detailed proof is omitted here. However, we need to emphasize that the proof requires the linear coefficients \((\alpha_c, \beta_c)\) to be the minimum MSE coefficients in estimating \(Y_2\) using \(Z_1\) and \(Z_{21}\), and \((\alpha_1^A, \beta_1^A, \gamma_1^A)\) (respectively, \((\alpha_2^A, \beta_2^A, \gamma_2^A)\)) to be the minimum MSE coefficients of estimating \(Y_1\) (respectively, \(Y_2\)) using \((Z_1, Z_{21}, Z_{22})\).

Proof of Theorem 1: Without loss of generality, we assume that \(\sigma_{y_1}^2 = \sigma_{y_2}^2 = \sigma_y^2\). Define \(d_1^x = \frac{d_1 x}{\sigma_y^2}\) and \(d_2^x = \frac{d_2 x}{\sigma_y^2}\). Then
\[
\beta_{\text{max}} = 1 + \sqrt{1 + \frac{4 \rho^2 d_2 d_1}{(1 - \rho^2)^2}}.
\]
Let \((Z_1, Z_{21}, Z_{22}, Q_1, Q_{21}, Q_{22})\) be the same random variables as in Lemma 3, such that
\[
E\left\{\frac{Q_1^2}{\sigma_y^2}\right\} = d_1 = \left(\frac{\beta_{\text{max}}}{2 \sigma_y^2} - \frac{1}{(1 - \rho^2)}\right)^{-1} \quad \text{(69)}
\]
\[
E\left\{\frac{Q_{21}^2}{\sigma_y^2}\right\} = d_{21} = \frac{\rho^2}{1 + d_1 (1 - 2 \rho^2)} - 1 \quad \text{(70)}
\]
\[
E\left\{\frac{Q_{22}^2}{\sigma_y^2}\right\} = d_{22} = \left(\frac{1}{d_2} - \frac{1}{d_2}\right)^{-1}, \quad \text{where} \quad d_2 = \left(\frac{\beta_{\text{max}}}{2 \sigma_y^2} - \frac{1}{(1 - \rho^2)}\right)^{-1} \quad \text{(71)}
\]
Then using (65) and (66) in Lemma 3, we have
\[
R_1 < I(Y_1; Z_1) - I(Z_1; Z_{21}) + \epsilon = R_{1}^* + \epsilon \quad \text{(72)}
\]
\[
R_2 < I(Y_2; Z_{21}) + I(Y_2; Z_{22}) - I(Z_{22}; Z_1 Z_{21}) + \epsilon = R_{2}^* + \epsilon \quad \text{(73)}
\]

The minimum MSE coefficients \((\alpha_c, \beta_c), (\alpha_1^A, \beta_1^A, \gamma_1^A), \text{ and } (\alpha_2^A, \beta_2^A, \gamma_2^A)\) are
\[
\alpha_c = \frac{\rho d_1}{\Omega}, \quad \beta_c = \frac{1 - \rho^2}{\Omega} + d_1, \quad \gamma_c = \frac{\rho d_1}{\Omega^*} + d_2, \quad \gamma_c = \frac{\rho d_1}{\Omega^*} + d_2 \quad \text{(74)}
\]
where \(\Omega = (1 + d_1)(1 + d_2 - \rho^2)\) and \(\Omega^* = (1 + d_1)(1 + d_2 - \rho^2)^{-1}\).

Thus, we can approach any point on the sum-rate bound (9).

Appendix B
Proof of Theorem 2

Proof: The proof of Theorem 2 is similar to that of Theorem 1, hence we only provide the necessary parameters. Denote \(d_i^x = \frac{d_i x}{\sigma_y^2}, n_1 = \sigma_{y_1}^2, n_2 = \sigma_{y_2}^2\), and define \((Q_1, Q_{21}, Q_{22})\) as in Lemma 3 such that
\[
E\left\{\frac{Q_1^2}{\sigma_y^2}\right\} = d_1 = \frac{2}{(d_i^x)^{-1} - 1 + n_1 n_1 - n_1} \quad \text{(77)}
\]
\[
E\left\{\frac{Q_{21}^2}{\sigma_y^2}\right\} = d_{21} = \frac{1}{1 + n_1 + d_1 (1 - 2 \rho^2)} - 1 - n_2 \quad \text{(78)}
\]
\[
E\left\{\frac{Q_{22}^2}{\sigma_y^2}\right\} = d_{22} = \left(\frac{1}{d_2} - \frac{1}{d_2}\right)^{-1}, \quad \text{where} \quad d_2 = \frac{2}{(d_i^x)^{-1} - 1 + n_2 n_2 - n_2} \quad \text{(79)}
\]
The minimum MSE coefficients are
\[
\alpha_c = \frac{d_{21}}{\Lambda}, \quad \beta_c = \frac{1 + n_2 (1 + n_1 + d_1) - 1}{\Lambda}, \quad \gamma_c = \frac{\alpha_c + d_2}{\Lambda^*}, \quad \gamma_c = \frac{n_1 + d_1}{\Lambda^*}, \quad \gamma_c = \frac{d_2}{d_2} \quad \text{(80)}
\]
where \(\Lambda = (1 + n_1 + d_1)(1 + n_2 + d_2) - 1\) and \(\Lambda^* = (1 + n_1 + d_1)(1 + n_2 + d_2) - 1\).
APPENDIX C
PROOF OF THEOREMS 3 AND 4

Proof: By setting $d_{21}$ in (70) to infinity, we can construct an asymmetric SWCQ coder $(\mathcal{E}_1^1, \mathcal{E}_2^1, \mathcal{D}_1)$ that achieves one corner point (denoted as $(R_1^1, R_2^1)$) of the sum-rate bound for the direct MT problem. On the other hand, by setting $d_{21}$ to $d_2$ in (71), we can construct another asymmetric SWCQ coder $(\mathcal{E}_1^2, \mathcal{E}_2^2, \mathcal{D}_1)$ that achieves the other corner point (denoted as $(R_1^2, R_2^2)$). Hence, any point on the sum-rate bound $\partial R_{12}^T(D^*)$ can be achieved by using time sharing between $(\mathcal{E}_1^1, \mathcal{E}_2^1, \mathcal{D}_1)$ and $(\mathcal{E}_1^2, \mathcal{E}_2^2, \mathcal{D}_1)$. This proves Theorem 3.

Similarly, by setting $d_{22}$ in (78) to infinity or to $d_2$ in (79), the two corner points of the sum-rate bound $\partial R_{12}^T(D^*)$ can be achieved. Hence, any point on $\partial R_{12}^T(D^*)$ can be achieved by time sharing, and Theorem 4 is proved.

APPENDIX D
PROOF OF LEMMA 1

Proof: First, we need to invert the regularity and symmetry conditions in designing a trellis $T$ (i.e., the corresponding convolution code $C$) [36].

1) Four costs $(D_0, D_1, D_2, D_3)$ should occur with equal frequency in the sense that
$$\sum_{i=1}^{N_d} \sum_{m=0}^{1} \chi(\phi(i, m) = (s, c)) = N_d/2, \quad c = 0, 1, 2, 3$$
(81)
where the indicator function $\chi = 1$ if the output part of the trellis mapping $\phi$ for state $i$ and input $m$ is $c$, and $\chi = 0$, otherwise.

2) Define $B_0 = D_0 \cup D_2$ and $B_1 = D_1 \cup D_3$, and denote the trellis output as $c = \phi(i, m)$; then for any $1 \leq i \leq N_d$, $D_0 \phi(i, 0) \cup D_3 \phi(i, 1)$ is either $B_0$ or $B_1$.

3) For any $1 \leq i \leq N_d$, let $j, k$ be the two distinct states satisfying $\phi(j, m_j) = i$ and $\phi(k, m_k) = i$, where $\phi(i, m)$ denotes the next-state part of the trellis mapping, then $D_0 \phi(j, m_j) \cup D_3 \phi(k, m_k)$ is either $B_0$ or $B_1$.

These conditions and the $\Sigma$-uniformity of $X$ ensure that each input vector $m$ (thus coset index vector $\mathbf{c}$) appears with equal probability, i.e., $P(C = T(m)) = P(M = m) = 2^{-m}$ for any $m \in \{0, 1\}^n$ (here the starting phase of TCQ is not considered). Hence, $P(\mathbf{X}_i \in \mathcal{D}_j) = \frac{1}{4}$ for $c = 0, 1, 2, 3$.

Note that the quantization noise $Q_i$ must be in the range $[-2, 2]$. For a given pair of $(q_i, x_i)$, since $q_i + x_i + v_i$ must be a signal point $j + 0.5$ with $j \in \mathbb{Z}$, $x_i + v_i$ can only take one value in the range $[x_i - 0.5, x_i + 0.5]$, i.e., $x_i + v_i = q_i + [x_i - q_i + 1] - 0.5$. Let $Y_i = X_i + V_i$, then $Q_i \rightarrow Y_i \rightarrow X_i$, hence

$$P(Q_i | Y_i, q_i | x_i) = \int_{y \in [x_i - 0.5, x_i + 0.5]} P(Q_i | Y_i = y | q_i | y) \cdot P(Y_i = y | x_i) \, dy$$
$$= \int_{y \in [x_i - 0.5, x_i + 0.5]} P(Q_i | Y_i = y | q_i) \, dy$$
$$= P(X_i \in D_0 | x_i = q_i + [x_i - q_i + 1] - 0.5)$$
$$= P(\mathbf{X}_i \in \mathcal{D}_0 | Y_i = q_i - 0.5)$$
(82)
which is independent of $x_i$.

The last equation of (82) is due to the following proposition, which states a key property of a nondithered trellis coded quantizer: statistical symmetry between cosets.

Proposition 1: Assume $f_X(x)$ is $\Sigma$-uniform with respect to $\mathcal{D}$ (with step size 1). Consider a trellis-coded quantizer $\mathcal{Q}_D^{\text{TCQ}}$ with $R_0 = 1$ and without dither. Let the quantized version of $X^n$ be $\mathbf{X}^n = [Q_D^{\text{TCQ}}(X)]^n$; then for sufficiently large $n$

$$P(X_i \in \mathcal{D}_c | x_i = q_i + [x_i - q_i + 1] - 0.5)$$
(83)
for $0 \leq i \leq n - 1, 0 \leq c \leq 3, j \in \mathbb{Z}, 2R + 1.5 \leq x_i, x_i + j \leq 2R - 1.5$.

Proof: First, consider the following two input vectors:
$$\mathbf{x} = \{x_1, x_2, \ldots, x_n\}, \mathbf{x}' = \{x_1 + 4i_1, x_2 + 4i_2, \ldots, x_n + 4i_n\}$$
(84)
where $i_j \in \mathbb{Z}$, and $2R + 1.5 \leq x_i, x_i + 4i_j \leq 2R - 1.5$, for $j = 1, 2, \ldots, n$. It is obvious that the Viterbi algorithm in TCQ produces the same coset index vector $\mathbf{c} = \mathcal{T}(\mathbf{M})$, and the codeword index vector of $\mathbf{x}'$ differs from that of $\mathbf{x}$ by $i = \{i_1, i_2, \ldots, i_n\}$. Consider the set $S_0^c = \{\mathbf{c} = \mathcal{T}(\mathbf{M}) : \mathbf{c} \in \{0, 1\}^n, c_j = j\}$ for $j = 0, 1, 2, 3$. Since $X$ is i.i.d., we have

$$P_c(\mathbf{c} | X_i = x_i) = P_c(\mathbf{c} | X_i = x_i + j)$$
$$\Rightarrow P(\mathbf{c} \in S_0^c | X_i = x_i) = P(\mathbf{c} \in S_0^c | X_i = x_i + j)$$
$$\Rightarrow P(\mathbf{X}_i \in \mathcal{D}_c | X_i = x_i) = P(\mathbf{X}_i \in \mathcal{D}_c | X_i = x_i + j)$$
(85)
for $0 \leq i \leq n - 1, 0 \leq c \leq 3, j \in \mathbb{Z}, 2R + 1.5 \leq x_i, x_i + j \leq 2R - 1.5$. Hence, we can assume that $x_i \in [0, 4]$ without loss of generality. Then $\Sigma$-uniformity implies that $X$ is uniformly distributed in $[0, 4]$.

Fix $c = 0$ and $j = 1$ with $i \gg 1$. We need to show that $P(\mathbf{X}_i \in \mathcal{D}_0 | X_i = x_i) = P(\mathbf{X}_i \in \mathcal{D}_0 | X_i = (x_i + 1) \mod 4)$ for any $x_i \in [0, 4]$. Let $\mathbf{c}' = \{c_1', c_2', \ldots, c_n'\} \in S_0^c$, then consider two input vectors $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and $\mathbf{x}' = \{x_1 + c_1', x_2 + c_2', \ldots, x_n + c_n'\} \mod 4$. Suppose $\mathbf{x}$ corresponds to a coset index vector $\mathbf{c}$, then $\mathbf{x}'$ must correspond to coset index vector $\mathbf{c}' = \mathbf{c} + \mathbf{e}'$ (and vice versa), where $\mathbf{c} + \mathbf{e}'$ denotes item-wise binary addition (XOR). Since the mapping $\mathbf{c} \rightarrow \mathbf{c} + \mathbf{e}'$ from $S_0^c$ to $S_0^c$ is one-to-one, it follows that

$$P(\mathbf{X}_i \in \mathcal{D}_0 | X_i = x_i) = \sum_{c \in S_0^c} P_c(\mathbf{e} | X_i = x_i)$$
$$= \sum_{c \in S_0^c} P_c(\mathbf{e} | X_i = (x_i + 1) \mod 4)$$
$$= P(\mathbf{X}_i \in \mathcal{D}_0 | X_i = (x_i + 1) \mod 4)$$. (86)

This result can be easily generalized to $c = 0, 1, 2, 3$ and $j = 1, 2, 3$. Thus, the proposition is proved.

APPENDIX E
PROOF OF THEOREM 5

Proof: Assume that Quantizer II in Fig. 5 is the dithered trellis-coded quantizer $\mathcal{Q}_21$ which uses an ESS of size $2R+1$, with $R_0 = 1$ and step size $\Delta_21$. Thus, the ESS $\mathcal{D} = \{-2R +$
\[ \Delta_{21}/2, -2^R + 3\Delta_{21}/2, \ldots, 2^R - \Delta_{21}/2 \} \] is partitioned into four cosets, each with \(2^R-1\) points. Then due to Proposition 1

\[
P(Y_{2,i} \in D_j | Y_{2,i} = y_{2,i}) = P(Y_{2,i} \in D_{(c+i) \mod 4} | Y_{2,i} = y_{2,i} + j\Delta_{21})
\]

for \(0 \leq i \leq n - 1\), \(0 \leq c, j \leq 3\), and \((-2^R + 1.5)\Delta_{21} \leq y_{2,i} + j\Delta_{21} \leq (2^R - 1.5)\Delta_{21}\). Denote the trellis bit vector of \(Q_{21}\) as \(m_{21} = \{m_{21,0}, m_{21,1}, \ldots, m_{21,n-1}\}\), and the codeword vector \(w_{21} = \{w_{21,0}, w_{21,1}, \ldots, w_{21,n-1}\}\). Now if we directly transmit the trellis bit vector \(m_{21}\) using one b/s (since \(\hat{R} = 1\)) without SW coding, the practical transmission rate \(R_{21}\) satisfies

\[
R_{21} = 1 + \frac{1}{n} H(W_{21} | M_{21}, V_{21}^n) = 1 + \frac{1}{n} H(W_{21} | C_{21}, V_{21}^n)
\]

\[
\leq 1 + \frac{1}{n} \sum_{i=0}^{n-1} H(W_{21,i} | C_{21,i}, V_{21,i})
\]

\[
= 1 + \frac{1}{n} \sum_{i=0}^{n-1} \Delta_{21}/2 \frac{1}{\Delta_{21}} H(W_{21,i} | C_{21,i}, V_{21,i} = v_{21,i}) dv_{21,i}.
\]

(87)

An example of the conditional distribution \(p(Y_{2,i} = y_{2,i} | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i})\) is shown in Fig. 10.

Next we consider the first WZ coding component which quantizes \(Y^n_1\) and compresses the quantization output \(I_1 = Q_1(Y^n_1)\) to \(R_1\) b/s. Let the ESS step size of the employed dithered TCO be \(\Delta_1\). Similar to (87) and (88), we have (89)–(90) shown at the bottom of the page, where \(V^n_1 = \{V_{1,i} \}_{i=0}^{n-1}\) is an i.i.d. random dither, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).

Here \(V_{21}^n = \{V_{21,i} \}_{i=0}^{n-1}\) is a length-\(n\) vector of i.i.d. random dithers, \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the coset index vector, and \(C_{21,i} = \frac{1}{n}M_{21,i}\) is the \(i\)th coset index for \(0 \leq i \leq n - 1\).
Fig. 11. Results of asymmetric SWCQ with TCQ and turbo/LDPC-based SW coding for the direct and indirect MT problems. The corner point with practical LDPC based SW coding is (2.602, 4.983) b/s, with a total sum–rate loss of 0.103 b/s. The corner point with practical turbo based SW coding is (2.273, 4.983) b/s, with a total sum–rate loss of 0.114 b/s. (a) Direct MT: $D_1^* = D_2^* = -30.00$ dB and $\rho = 0.99$; (b) Indirect MT: $D_1^* = -22.58$ dB and $\sigma_{r_1}^2 = \sigma_{r_2}^2 = \frac{1}{0.97}$.

length-$n$ vector of i.i.d. random dithers, and ($\star$) is true since the Markov chain $\hat{Z}_{21;1,1} \rightarrow Y_{1;1,1} \rightarrow C_{1,1}$ holds.

Similar results can be obtained for the second WZ coding component which quantizes $Y_{2;2}^*$ and compresses the quantiza-
tion output $I_{22} = Q_{22}(Y_2^2)$ to $R_{22}$ b/s; see (91)–(92) at the bottom of the page, where $V_{22}^n = \{V_{22,i}^n\}_{i=0}^{n-1}$ is a length-$n$ vector of i.i.d. random dithers, and $(\ast)$ is true since the Markov chain $\hat{Z}_{ci} \rightarrow Y_{2,i} \rightarrow C_{2,i} \rightarrow \hat{Z}_{2,i}$ holds.

Equations (87)–(92) are based on the assumption of $\Sigma_t$-uniformity and are very difficult to compute in practice. However, at high rate, all the TCQ step sizes $\Delta_{21}, \Delta_1, \Delta_{22}$ tend to zero. Thus (see Fig. 10)

$$p(W_{21,i} = j | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i}) = p(Y_{2,i} + v_{21,i} \in \mathcal{W}_j | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i}) \approx p(Y_{2,i} + v_{21,i} \in \mathcal{W}_j)$$

(93)

where

$$\mathcal{W}_j = [(4j + c_{21,i} - 2n^2, 1) \Delta_{21}, (4j + c_{21,i} - 2n^2, 1) \Delta_{21}]$$

Then we get (94) at the bottom of the page, where “$A \leadsto B$” means “$A$ approaches $B$ asymptotically,” or $\lim_{\Delta_1 \rightarrow 0, \Delta_2 \rightarrow 0, \Delta_{22} \rightarrow 0} [A - B] = 0$.

On the other hand, assuming ideal SW coding in the sense that $\hat{Z}_{21,i} = Z_{21,i}$, due to the definition of normalized second moment $G_{Q_{21}}, \tau$, we have

$$d_{21} = \frac{1}{n} E(||Z_{21,i}^2 - Y_2^2||^2) \approx \frac{1}{n} E(||Z_{21,i}^2 - Y_2^2||^2) = V/4G_{Q_{21}} = (2\Delta_{21})^2/4G_{Q_{21}},$$

(95)

Hence

$$R_{21} = 1 + \frac{n-1}{n} \sum_{i=0}^{n-1} \frac{\Delta_{21}}{2} \frac{1}{\Delta_{21}} H(W_{21,i} | C_{21,i}, V_{21,i} = v_{21,i}) dv_{21,i}$$

$$\approx 1 + \frac{n-1}{n} \sum_{i=0}^{n-1} h(Y_{2,i} + v_{21,i}) - \log(4\Delta_{21})$$

$$\approx \frac{1}{2} \log(\frac{\sigma_{21}^2}{d_{21}}) + \frac{1}{2} \log(2\pi e G_{Q_{21}}).$$

(96)

Similarly, we write

$$p(W_{1,i} = j | C_{1,i} = c_{1,i}, \hat{Z}_{21,i} = \hat{z}_{21,i}, V_{1,i} = v_{1,i})$$

$$= p(Y_{1,i} + v_{1,i} \in \mathcal{W}_j | C_{1,i} = c_{1,i}, \hat{Z}_{21,i} = \hat{z}_{21,i}, V_{1,i} = v_{1,i})$$

$$= \frac{1}{\mathcal{W}_j} \int_{\mathcal{W}_j} p(Y_{1,i} + v_{1,i} = \tau | \hat{Z}_{21,i} = \hat{z}_{21,i}) \cdot P(C_{1,i} = c_{1,i} | Y_{1,i} = \tau) d\tau$$

$$= p(Y_{1,i} + v_{1,i} = \tau | \hat{Z}_{21,i} = \hat{z}_{21,i}) \int_{\mathcal{W}_j} \frac{P(C_{1,i} = c_{1,i} | Y_{1,i} = \tau)}{P(C_{1,i} = c_{1,i} | \hat{Z}_{21,i} = \hat{z}_{21,i})} d\tau$$

$$\approx p(Y_{1,i} + v_{1,i} = \tau | \hat{Z}_{21,i} = \hat{z}_{21,i}) \int_{\mathcal{W}_j} \frac{P(C_{1,i} = c_{1,i} | Y_{1,i} = \tau)}{P(C_{1,i} = c_{1,i} | \hat{Z}_{21,i} = \hat{z}_{21,i})} d\tau.$$

(97)

Then we get (98), also at the bottom of the page. Hence

$$R_1 = 1 + \frac{n-1}{n} \sum_{i=0}^{n-1} \frac{\Delta_{21}}{2} \frac{1}{\Delta_{21}} H(W_{1,i} | C_{1,i}, \hat{Z}_{21,i}, V_{1,i} = v_{1,i}) dv_{1,i}$$

$$\approx \frac{1}{2} \log(\frac{\sigma_{12}^2}{d_{12}}) + \frac{1}{2} \log(2\pi e G_{Q_{21}}).$$

(99)

$$R_{22} \leq 1 + \frac{n-1}{n} \sum_{i=0}^{n-1} \frac{\Delta_{22}}{2} \frac{1}{\Delta_{22}} H(W_{22,i} | C_{22,i}, \hat{Z}_{ci,i}, V_{22,i} = v_{22,i}) dv_{22,i},$$

(91)

$$p(Y_{2,i} = y_{2,i} | C_{22,i} = c_{22,i}, \hat{Z}_{ci,i} = \hat{z}_{ci,i}, V_{22,i} = v_{22,i})$$

$$= p(Y_{2,i} = y_{2,i} + v_{22,i} | \hat{Z}_{ci,i} = \hat{z}_{ci,i}) \cdot P(C_{22,i} = c_{22,i} | Y_{2,i} = y_{2,i} + v_{22,i}, \hat{Z}_{ci,i} = \hat{z}_{ci,i})$$

$$\approx p(Y_{2,i} = y_{2,i} + v_{22,i} | \hat{Z}_{ci,i} = \hat{z}_{ci,i}) \cdot P(C_{22,i} = c_{22,i} | Y_{2,i} = y_{2,i} + v_{22,i}),$$

(92)

$$H(W_{21,i} | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i})$$

$$= - \sum_{j=0}^{2^n-1} p(W_{21,i} = j | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i}) \log p(W_{21,i} = j | C_{21,i} = c_{21,i}, V_{21,i} = v_{21,i})$$

$$\approx h(Y_{2,i} + v_{21,i}) - \log(4\Delta_{21})$$

$$\approx h(Y_{2,i}) - \log(4\Delta_{21}),$$

(94)

$$H(W_{21,i} | C_{21,i} = c_{21,i}, \hat{Z}_{21,i}, V_{1,i} = v_{1,i})$$

$$= - \int_{R} \sum_{j=0}^{2^n-1} [p(W_{21,i} = j | C_{21,i} = c_{21,i}, \hat{Z}_{21,i} = \hat{z}_{21,i}, V_{1,i} = v_{1,i}) \log p(W_{21,i} = j | C_{21,i} = c_{21,i}, \hat{Z}_{21,i} = \hat{z}_{21,i}, V_{1,i} = v_{1,i})] d\hat{z}_{21,i}$$

$$\approx - \int_{R} \sum_{j=0}^{2^n-1} [p(Y_{1,i} + v_{1,i} \in \mathcal{W}_j | \hat{Z}_{21,i} = \hat{z}_{21,i}) \log p(Y_{1,i} + v_{1,i} \in \mathcal{W}_j | \hat{Z}_{21,i} = \hat{z}_{21,i})] d\hat{z}_{21,i}$$

$$\approx h(Y_{1,i} + v_{1,i} | \hat{Z}_{21,i}) - \log(4\Delta_{1})$$

$$\approx h(Y_{1,i} | \hat{Z}_{21,i}) - \log(4\Delta_{1}).$$

(98)
Fig. 12. Results of symmetric SWCQ with TCQ and turbo/LDPC-based SW coding for the direct and indirect MT problems. The corner point with practical LDPC based SW coding is \((2.3, 4.979)\) b/s, with a total sum-rate loss of 0.157 b/s. The corner point with practical turbo based SW coding is \((2.315, 4.979)\) b/s, with a total sum-rate loss of 0.152 b/s. (a) Direct MT: \(D_1^* = D_2^* = -30.00\,\text{dB}\) and \(\rho = 0.99\). (b) Indirect MT: \(D^* = -22.58\,\text{dB}\) and \(\sigma_{\delta_1}^2 = \sigma_{\delta_2}^2 = \frac{1}{N_0}\).
Similarly, $R_{22}$ can be written as

$$R_{22} = 1 + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{Z_{ii}} H(W_{2i}, |C_{2i}, Z_{ii}, V_{i}, \tilde{Z}_{ii}, \bar{V}_{2i} \bar{Z}_{2i}) dv_{2i}, $$

$$\approx \frac{1}{2} \log\left(\frac{1}{a_{22}}\right) + \frac{1}{2} \log(2\pi e G_{Q22}).$$

(100)

Finally, due to (72) and (73) in the proof of Theorems 1

$$\frac{1}{2} \log\left(\frac{1}{a_{21}}\right) \approx R_{1}^{*}, $$

$$\frac{1}{2} \log\left(\frac{1}{a_{22}}\right) + \frac{1}{2} \log(2\pi e G_{Q22}) \approx R_{2}^{*}. $$

(101)

Therefore, (40) is true and the theorem proved.

APPENDIX F

PROOF OF THEOREM 6

Proof: At high rate, there is no loss in transmitting the trellis bit-planes $J_{1}^{1}$ and $K_{1}^{1}$ using 2 b/s. Then the total rate transmission of our symmetric SWCQ scheme is

$$2 + \frac{1}{n} H(W_{1}, W_{2} | C_{1}, C_{2}, V_{1}, \bar{V}_{2})$$

b/s. Now let $(R_{1}^{*}, R_{2}^{*})$ be one corner point of the sum-rate bound. By setting $d_{22}$ to infinity, we have

$$R_{21} + R_{1}^{*} = \left[ 1 + \frac{1}{n} H(W_{1} | C_{1}, V_{1}) \right] + \left[ 1 + \frac{1}{n} H(W_{2} | C_{2}, \bar{Z}_{2}, V_{1}^{n}) \right]$$

$$= 2 + \frac{1}{n} H(W_{1}, W_{2} | C_{1}, C_{2}, V_{1}, \bar{V}_{2})$$

$$= R_{1}^{*} + R_{2}^{*} + \frac{1}{2} \log(2\pi e G_{Q1}) + \frac{1}{2} \log(2\pi e G_{Q2}) + o(1)$$

$$= R_{1}^{*} + R_{2}^{*} + \frac{1}{2} \log(2\pi e G_{Q1}) + \frac{1}{2} \log(2\pi e G_{Q2}) + o(1).$$

(102)

Then the theorem readily follows.

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References


