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Pricing the Default Option of Inflation-Indexed Mortgages Using Explicit Finite Difference Method*

Işıl Erol
Department of Economics, Middle East Technical University, Ankara-Turkey;
E-mail: isile@metu.edu.tr; Tel: 90-312-2103048; Fax: 90-312-2107964

Kanak Patel
Department of Land Economy, University of Cambridge, England;
E-mail: kp10005@cam.ac.uk; Tel: 44-1223-332153; Fax: 44-1223-337130

This paper evaluates the default risk of civil servants' wage-indexed payment mortgage (WIPM) contract in Turkey, which is linked to the expected inflation. The aim of the study has two sides: one is to apply the contingent claims approach, which has been widely used to price standard fixed- and adjustable-rate contracts, to price an inflation-indexed mortgage. The second is to understand if WIPM contract is a suitable mortgage design for lenders under an inflationary economy. We extend the traditional risk-neutral valuation for pricing the WIPM contract with its embedded default option. Using backward pricing method, namely the explicit finite difference method, we evaluate this unique inflation-indexed mortgage contract from the lender's point of view. The expected inflation and house price are the two stochastic variables underlying the WIPM contract. Our numerical results show that the lender benefits from originating WIPM only during the periods when the real interest rate is very low. Expected inflation risk premium notably increases the value of future payments on WIPM contract, resulting in high values of lender's position in the mortgage agreement. The results also show that house price volatility has a greater effect on the borrower's default option value compared to the expected inflation volatility

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Keywords

option pricing; inflation-indexed mortgages; default option; explicit finite difference method; wage-indexed payment mortgage (WIPM)

Introduction

In emerging economies, the importance of index-linked mortgages in facilitating the long-term mortgage lending and borrowing has been widely recognized. Due to highly volatile economic conditions, inflation-indexed mortgages, such as price level adjusted mortgage (PLAM) and indexed units of account (UDI) mortgage, and dual-index mortgage (DIM) contract have become more popular in comparison to the standard mortgage contracts. Inflation uncertainty increases the risk in nominal contracts, and consequently decreases the real rate of return for the mortgage lenders. In high inflation economies, such as Mexico, Chile, Brazil, Argentina, Turkey, etc. it is essential for mortgage lenders to design suitable mortgage contracts that provide hedges against unexpected inflation.

The purpose of this study is to evaluate the civil servants' wage-indexed payment mortgage (WIPM) contract, an alternative inflation-indexed mortgage that was widely originated in Turkey during the late 1990s inflationary period. Berument and Gunay (2001) report that studying the effect of inflation uncertainty is highly important in Turkey, where high and variable inflation rates over more than two decades provide a laboratory environment. After a decade of high inflation (positive unexpected inflation) between 1987 and 1997, the leading mortgage lender in Turkey originated the WIPMs. This contract provides an ideal test case for evaluating the inflation hedging performance of an index-linked mortgage contract in an environment where the interest rate is highly volatile. In this paper, we propose a valuation model for the WIPM contract based on the contingent claims approach. We demonstrate how the traditional risk-neutral valuation principle can be extended in order to evaluate this specific type of inflation-indexed mortgage with its embedded default option from the lender's perspective.

Against the backdrop of persistent inflationary pressure, the Turkish government in 1998 embarked upon a major housing finance reform. The government in collaboration with a state-owned bank (Emlak Bank) designed the WIPM contract. The WIPM is based upon the civil servant's wage (CSW) index, which is linked to the expected inflation. In a typical WIPM contract, there is no contracted mortgage rate, no periodic or lifetime cap that constrains the payment adjustments and no pre-determined margin to be added to the CSW index. Emlak Bank created this specific mortgage design for middle-income civil servants, who are the main group of borrowers of housing loans with their state-guaranteed salaries. And the government introduced a policy to link the CSW rate to the expected inflation. The aim of this policy was to facilitate mortgage financing to an important sector of the population, namely the middle-income public sector employees.

Examining the mortgage markets in emerging economies and the performance of their mortgage products, the recent studies mainly focus on the Mexican mortgage market. Lea and Bernstein (1996) attempt to demonstrate how mortgage instruments in the Mexican mortgage market can affect the institution providing the loans. Lipscomb and Hunt (1999) examine the mechanics and the behaviour of UDI mortgages, which are the price-level adjusting mortgages, in comparison to the DIMs. Lipscomb et al. (2003) analyse the exchange-rate risk of price-level adjusting mortgages. The authors show that UDI mortgages reduce exchange-rate risk for mortgage lenders that have foreign capital as their source of funding.

Pickering (2000) uses Monte Carlo simulation method in order to estimate the performance of DIMs originated by SOFOLES, which are the newly created financial intermediary to maintain mortgage lending to low-income households in Mexico. The author concludes that these financial institutions are positioned to become new leaders in the Mexican mortgage market. A few studies analyse the mortgage markets in transition economies. Jaffee and Renaud (1997) discuss the main factors that hinder the development of mortgage markets in economies that are in transition from central planning to a market system. Analysing the Polish mortgage market, Chiquier (1998) compares the performance of DIM with the standard fixed-and variable rate mortgage instruments. Chiquier shows that in unstable economic conditions in Poland fixed-rate, long-term mortgages create large interest rate risk for lenders and an affordability problem for borrowers. Variable rate mortgages result in excessive initial payments and later insignificant ones.

Index-linked mortgages, particularly the PLAM, were also used in the US mortgage market during the inflationary periods of early-1970s. Several studies have argued the case for PLAM as an alternative mortgage design to adjustable-rate mortgage (ARM) and have analysed PLAM in comparison with the standard fixed-rate mortgage (FRM) and ARM contracts. See Cohn and Fischer (1975), Kearl (1979), Baesel and Biger (1980), Statman (1982), Scott et al. (1993), Elmer (1992), McCulloch (1986) and Kim (1987).

In contrast to the commonly used Monte Carlo simulation methodology to estimate the performance of index-linked mortgages, this study uses contingent claims approach in order to evaluate the WIPM contract, which is a particular inflation-indexed mortgage. Contingent claims approach has been used extensively for pricing the standard FRM. Some authors have also focused on pricing ARM using contingent claims approach. Ramaswamy and Sundaresan (1986) use a backward method to evaluate floating rate notes whose coupon is dependent on the entire history of interest rates. Kau et al. (1990, 1993) use the explicit finite difference method in order to price default-free and defaultable ARMs, respectively. Stanton and Wallace (1995, 1999) use the Crank-Nicholson finite difference approximation to value ARMs with prepayment option.

To the best of our knowledge, this study is the first attempt to evaluate an inflation-indexed mortgage contract using the standard contingent claims approach. WIPMs are treated as derivative assets whose prices depend upon the evolution of house prices and the civil servant's wage rate (the expected inflation). Using backward pricing method, namely explicit finite difference method, we price the WIPM contract and its embedded option to default. We show that it is highly profitable for the lender to originate mortgages that are indexed to expected inflation, rather than highly volatile interest rate, when the real interest rate decreases and becomes negative due to positive unexpected inflation. We believe that our research has important implications for other developing mortgage markets, where the inflation uncertainty is high.

The remainder of the paper proceeds as follows. The next section explains the WIPM contract details and outlines the Turkish government's housing policy for financing the public sector housing. In the third section we describe the basic valuation model of the WIPM contract against the background of the economic environment, and then pricing the WIPM contract with its embedded default option. This section also presents the numerical solution of the WIPM valuation model. The fourth section provides an analysis of the numerical results, and the final section presents some concluding remarks.

Wage-Indexed Payment Mortgage (WIPM) Contract

The WIPM has a ten-year mortgage term with an initial maximum loan-to-value ratio of 75%. Mortgage repayments are indexed to a measure of income in order to maintain the affordability of the loan to the household income. Because the repayments can vary, the loan term must also be variable to accommodate shortfalls in payments when wages are changing rapidly.

WIPM differs significantly from the ARM by having no contracted mortgage rate. This mortgage instrument does not have periodic or lifetime caps that constrain the payment adjustments, and a pre-specified margin to be added to the current value of the CSW index. Also, there is no arrangement fee that is charged to the borrowers at loan origination. The adjustment amount of the outstanding WIPM balance, at a given semi-annual adjustment date, is calculated by multiplying outstanding balance by the civil servant's wage rate (CSWR), which is the percentage change in the CSW index. At the beginning of every January and July, the government announces the expected inflation and the Ministry of Finance sets the CSWR in line with the expected inflation over the next six months. That is,

$$CSWR_{t+1} = {}_t\pi^{e}_{t+1} \tag{1}$$

The actual inflation at a semi-annual date at time $t+1$, π^{a}_{t+1}, may be higher or lower than the government's announced expected inflation at time t, that is

$$\pi^{a}_{t+1} = {}_t\pi^{e}_{t+1} + \varepsilon_{t+1} \tag{2}$$

where ε_{t+1} is the unexpected inflation and $E\left[\varepsilon_t\right] = 0.017$ for the sample period of 1994 to 2004.

$$\text{If } \varepsilon_{t+1} > 0 \qquad CSWR_{t+1} < \pi^{a}_{t+1} \tag{3a}$$

$$\text{If } \varepsilon_{t+1} < 0 \qquad CSWR_{t+1} > \pi^{a}_{t+1} \tag{3b}$$

If actual inflation during the semi-annual period $t+1$ is higher than the officially expected rate announced at time t for period $t+1$, the government pays out to civil service employees in cash the difference plus an additional fixed mark-up of 2%. For example, in January 1999 the expected inflation, $\pi^{e}_{99/1}$ = 0.3, the CSW is also 0.3 (see Figure 1). Since the actual inflation in July 1999 $\pi^{a}_{99/1}$ = 0.35, the civil service employees received a cash compensation equivalent to 0.05 increase in their wage, plus an additional fixed mark-up of 0.02 known as a welfare share, in July 1999. This unanticipated positive inflation compensation in July 1999 is for only the semi-annual period between January and July 1999 and it does not affect any future CSWR. Although the government adjusts the civil service employees' wage rate, the mortgage repayment ($\mathrm{MP}_{99/2}$) is calculated based on the expected inflation only, which is then fixed for the next six months. Thus,

$$\mathrm{MP}_{t+1} = f\left(\mathrm{CSWR}_{t+1}\right) \tag{4}$$

or

$$\mathrm{MP}_{t+1} = f\left({}_{t}\pi^{\mathrm{e}}_{t+1} \right).$$

During the period 1994 and 2004, the house price index (HI) and consumer price index (CPI) were highly correlated, with the correlation coefficient of 87.6%. Thus, changes in the house price index tracks the movements in the actual inflation (see Figure 2). That is,

$$\Delta\,\mathrm{HI} = \left(\mathrm{HI}_{t+1} - \mathrm{HI}_{t}\right) / \,\mathrm{HI}_{t} \cong \pi^{\mathrm{a}}_{t+1} \tag{5}$$

$$\text{If } \varepsilon_{t+1} > 0\,,\ \Delta\mathrm{HI} \cong \pi^{\mathrm{a}}_{t+1} > {}_{t}\pi^{\mathrm{e}}_{t+1} \tag{6a}$$

$$\text{If } \varepsilon_{t+1} < 0\,,\ \Delta\mathrm{HI} \cong \pi^{\mathrm{a}}_{t+1} < {}_{t}\pi^{\mathrm{e}}_{t+1} \tag{6b}$$

Under the circumstances that six-month cumulative value of actual inflation outpaces the expected inflation, and therefore the CSW index, (see Equation (6a)), there is no incentive for borrowers to default on their mortgages. This is because, firstly, their outstanding debt amount is adjusted to the CSW rate, which is lower than the actual inflation rate, and secondly, the percentage increase in HI is greater than the CSW rate. However, for the lender the real return is negative when $\pi^{\mathrm{a}}_{t+1} > {}_{t}\pi^{\mathrm{e}}_{t+1}$. This is precisely what happened in the first half of 2001 when the expected inflation rate ${}_{00/2}\pi^{\mathrm{e}}_{01/1}$, and so the $\mathrm{CSWR}_{01/1}$, was set at 15.9%, while the actual inflation rate $\pi^{\mathrm{a}}_{01/1}$ was 32.32% (see Figure 1). Conversely, if the actual inflation is lower than the expected inflation rate (see Equation (6b)), the lenders realise an unexpected gain. However, a lower value of π^{a} increases the borrowers' incentive to default on their mortgages because they bear the burden of a considerably higher amount of mortgage repayment at a time when house price index has declined sharply. This was actually the case in the first half of 2002, when the expected inflation rate ${}_{01/2}\pi^{\mathrm{e}}_{02/1}$, and also $\mathrm{CSWR}_{02/1}$, was set as 27.68% while the actual inflation rate $\pi^{a}_{02/1}$ was 12.09% (see Figure 1).

The WIPM contract design is similar to that of the PLAM and DIM, which is based on an indexation formula that amortizes the loan balance. However, WIPM contract does not have either nominal or real amortization rate (See Appendix 1 for monthly repayment modelling of WIPM contract). This design has evolved specifically because the interest rate in Turkey has been highly volatile over the last fifteen years.

Figure 1: Semi-annual changes in CSW index and consumer price index (CPI) between 1994 and 2004

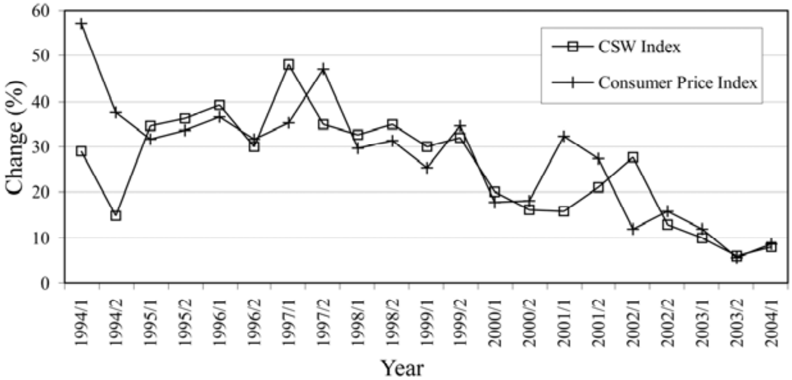
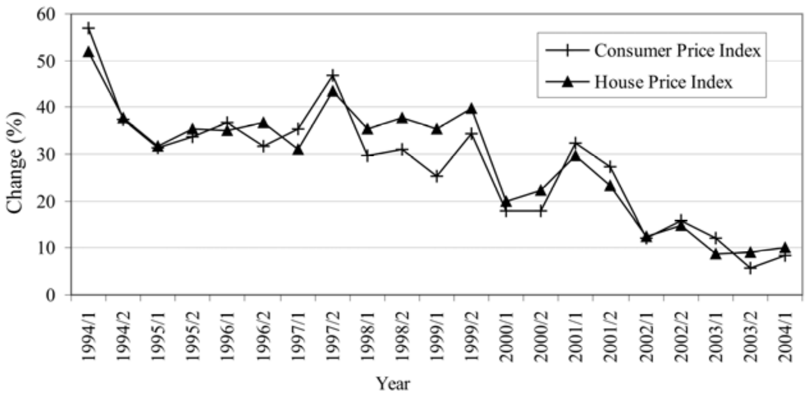


Figure 2: Semi-annual changes in house price index (HPI) and consumer price index (CPI) between 1994 and 2004



* **Source**: Consumer price index (CPI) and house price index (HPI) data are obtained from State Institute of Statistics and Civil Servants' Wage (CSW) Index data are obtained from the Ministry of Finance.

A recent study by Berument and Malatyali (1999) analyses the behaviour of the Turkish Treasury interest rates based on the Fisher hypothesis. This study uses the sample period from November 1988 to June 1998. In their regression of interest rate on expected and unexpected inflation, the authors find that both coefficients of expected inflation and inflation risk are statistically significant. The empirical findings reveal that while the interest rate is positively related to expected and unexpected inflation, the interest rate increases less than expected inflation.[1] This empirical evidence supports Tobin's (1965) hypothesis, during periods of high inflation (due to positive unexpected inflation) the real interest rate declines. The observed real interest rate has even become negative in Turkey. It can be seen from Figure 3 that between 1987 and 1998 the real interest rate was on average negative.[2]

In Turkey, nominal mortgage contracts such as DIM and ARM would have resulted in payment shocks for borrowers leading to default risk, and would have produced negative real return for lenders, leading to real interest rate risk. Mortgages indexed to the expected inflation provide a protection against high mortgage defaults. Lenders also benefit from originating mortgages indexed to the expected inflation, rather than highly volatile nominal market interest rate, as long as the real interest rate declines in high inflationary conditions. The PLAM and UDI mortgage contracts, which are fixed real rate loans, also insulate both the borrower and lender in real terms from the volatile interest rates. However, these mortgage designs suffer from major payment shocks in that if the inflation rate rises faster than wage rates for any period of time, the payment burdens of the borrowers can become

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[1] Berument and Malatyali (1999) specify that the inflation rate follows an autoregressive process in order q. That is $\pi_t^a = i_0 + \sum_{j=1}^{q} i_j \pi_{t-j}^a + \varepsilon_t$

The conditional expectation of the inflation rate at time t with the given information set at time $t-1$ is $E\left(\pi_t / \Omega_{t-1}\right) = i_0 + \sum_{j=1}^{q} i_j \pi_{t-j}$

The authors use ARCH model in order to forecast the inflation risk or conditional variance of unanticipated inflation at given time t as $h_t^2 = c_0 + \sum_{j=1}^{p} c_{1j} \varepsilon_{t-j}^2$

In order to capture the effect of positive unexpected inflation on interest rate, the Fisher equation is modified to include inflation risk, conditional standard deviation of unexpected inflation, h_t in Quasi Maximum Likelihood regression analysis. The estimates of the modified Fisher equation are the following, where t-statistics are reported in parentheses:

$r_t = 0.032 + 0.55\pi_t^e + 0.94h_t$ (4.18) (2.26) (2.46), $R^2 = 0.89$

[2] Berument and Malatyali (1999) use the treasury interest rates in their regression analysis; however, the Turkish government did not continuously use the treasury bills. The International Financial Statistics (IFS) data on T-Bill rates are from February 1994 to January 1996 and from January 1999 to December 2002. Therefore, the authors use interest rates of the bills traded in the secondary markets and assume that these bills are held for 1 month and the real interest rates are realised at the end of that period. Using the IFS data on money market interest rates and inflation (CPI) data, reported by the State Institute of Statistics, we calculated the real interest rates both on annual and monthly basis (see Figures 3a and 3b). On average, the observed real interest rates have been negative between 1987 and 1997, which supports Berument and Malatyali's research that high inflation results in declining and even negative real interest rates in Turkey.

unsustainable, resulting in a high level of defaults.

Figure 3a: Market interest rate and inflation (consumer price index) movements in Turkey between 1990 and 2002 (monthly interval)

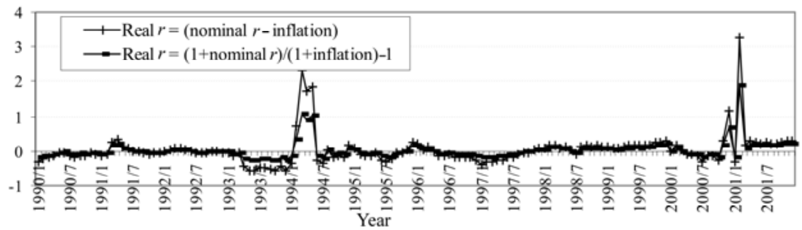
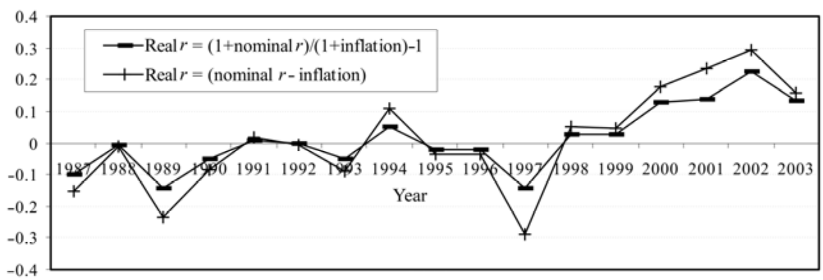


Figure 3b: Annual real interest rates in Turkey between 1987 and 2003



* **Source**: Inter-bank Money Market Rates are obtained from International Financial Statistics (IFS) prepared by the International Monetary Fund (IMF) and Consumer Price Index (CPI) data are obtained from State Institute of Statistics.

Valuation of the WIPM Contract

In standard ARM and FRM contracts the periodical coupon payments are based on the risk-free interest rate. The fair value of the mortgage loan is defined as that value, which makes the present value of expected future income on the housing property, discounted at the mortgage coupon rate, equal to the original loan amount. The time value of expected mortgage cash flows is calculated using the appropriate market interest rate. This measure reflects the market price of the mortgage loan rather than lender's personal preferences. However, the individual lender may of course disagree with these market prices and view some loans to be under-or overvalued in the market (Tuckman, 1995). Emlak Bank, which has dominated the Turkish mortgage market since the early 1990s, has been using the CSW rate that is linked to the expected inflation, rather than the market interest rate to determine the periodical repayments on WIPM contract.

Economic environment

WIPMs are effectively derivative assets whose prices depend on the evolution of the CSW rate and house price index. In our WIPM valuation model, following the standard contingent claims approach, the value of the house is assumed to follow the standard geometric Brownian motion (GBM) process. The GBM process implies that house prices have a lognormal diffusion process shown in Equation (7). The GBM process implies that house prices have a lognormal diffusion process shown in Equation (7). The return to owning the housing property consists both of price appreciation and of a service flow. Since the householder receives benefit from living in the house, the term s is included to denote the constant rate of service flow, or value of implicit rent, from the house (see Kau et al., 1993 and 1995).

$$\frac{\mathrm{d}H}{H} = \left(\mu - \mathrm{s}\right)\mathrm{d}t + \sigma_H \mathrm{d}z_H \tag{7}$$

where μ denotes the instantaneous average rate of house price appreciation and σ_H represents the volatility of disturbances in actual house price appreciation around the trend rate $\left(\mu - s\right)$, and z_H is the standardized Wiener process that drives the uncertainty in house prices. The specific mechanism used for the CSW rate dynamics is

$$\mathrm{d}w = \kappa\left(\theta_w - w\right)\mathrm{d}_t + \sigma_w \sqrt{w}\mathrm{d}z_w \tag{8}$$

This is the mean-reverting square root diffusion process, or the CIR model, where θ_w represents the long-term mean value for the changes in CSW index, κ is the speed of adjustment in the mean reverting process, σ_w denotes the instantaneous standard deviation of the wage rate disturbance, w is the wage rate, and z_w is the standardized Wiener.[3] The unanticipated

[3] The mean reverting process (MRP) has been used extensively in the valuation models for interest rate sensitive and inflation, CPI, sensitive contingent claims. According to Buetow and Albert (1998), MRP processes are appropriate for positive economic variables that tend toward a long-run mean but experience short-term disturbances, so they are often used to model interest rates and the inflation rate. The second reason for choosing the mean reverting square root process for the CSW rate is that an increase in wage rates can be interpreted as a yield on human capital. In a two-factor oil contingent claims pricing model, Gibson and Schwartz (1990) use spot price of oil and convenience yield on crude oil, which is assumed to follow a mean reverting process. They view the convenience yield as a net dividend yield accruing to the owner of the physical commodity of crude oil. Analogously, for pricing wage indexed mortgage contracts, the wage level can be defined as the yield on human capital.

change in the value of house is assumed to be correlated with the unanticipated change in wage rate. Thus,

$$\mathrm{d}z_H(t)\,\mathrm{d}z_w(t) = \rho \mathrm{d}t \tag{9}$$

where ρ denotes the instantaneous correlation coefficient between the Wiener processes. It is clear from Equation (8) that the expected growth rate in w is different from the risk-free interest rate, reflecting Emlak Bank's risk preferences. However, there are many different risk-neutral worlds that can be assumed in any given situation. Following Hull (2003), we define the market price of expected inflation risk in order to derive the risk-neutral valuation. Consider the properties of derivative dependent on the value of a single variable, θ, which has a stochastic process of $\frac{\mathrm{d}\theta}{\theta} = m\mathrm{d}t + v\mathrm{d}z$, where m is the expected growth rate in θ, and v is the volatility of θ. The market price of risk of the variable θ, λ_θ, is given by

$$\lambda_\theta = \frac{m-r}{v} \tag{10}$$

If f is the asset price, which is dependent upon the underlying variable θ, and time, t, the following partial differential equation (PDE) must be satisfied by f [4]

$$\frac{\partial f}{\partial t} + \left(m - \lambda_\theta v\right)\theta\frac{\partial f}{\partial \theta} + \frac{1}{2}v^2\theta^2\frac{\partial^2 f}{\partial \theta^2} = rf \tag{11}$$

Interpreting the risk neutrality in pricing bonds, Wilmott (2000) also uses Equation (11) and states that when pricing interest rate derivatives it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$\mathrm{d}r = \left(u - \lambda w\right)\mathrm{d}t + w\mathrm{d}z_r \tag{12}$$

when the real spot rate is governed by the stochastic differential equation of the form $\mathrm{d}r = u\left(r,t\right)\mathrm{d}t + w\left(r,t\right)\mathrm{d}z_r$. We know that the general Black-

[4] The asset price f follows a process of the form $\mathrm{d}f = \mu f\mathrm{d}t + \sigma f\mathrm{d}z$, where $\mu - r = \lambda\sigma$. Since f is a function of θ and t, we can use Ito's lemma to express μ and σ in terms of m and v. The result is

$$\mu f = m\theta\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial t} + \frac{1}{2}v^2\theta^2\frac{\partial^2 f}{\partial \theta^2} \text{ and } \sigma f = v\theta\frac{\partial f}{\partial \theta} \text{ (see Hull, 2003).}$$

Scholes PDE for a dividend paying stock is

$$\frac{\partial f}{\partial t} + \left(r - q\right) S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{13}$$

where S is the stock price and q gives the dividend yield rate. Comparing Equation (11) with Equation (13), we see that the differential equation for the price of an asset dependent on θ is the same as that for a derivative dependent on an asset providing a dividend yield equal to q, where $q = r - m + \lambda_\theta v$. This observation leads to a way of extending the traditional risk-neutral valuation result. Any solution to Equation (13) for S is a solution to Equation (11) for θ, and vice versa, when the substitution $q = r - m + \lambda_\theta v$ is made. Thus, we can solve Equation (11) by setting the expected growth of θ equal to $r - \left(r - m + \lambda_\theta v\right) = m - \lambda_\theta v$ and discounting expected payoffs at the risk-free rate. In our WIPM valuation model setting the return from the asset dependent on w to the risk-free rate of interest, the stochastic process for w, Equation (8) becomes[5]

$$\mathrm{d}w = \left[\kappa(\theta_w - w) - \lambda_w \sigma_w \sqrt{w}\right]\mathrm{d}t + \sigma_w \sqrt{w}\mathrm{d}z_w \tag{14}$$

$m = \kappa(\theta_w - w)$ = the expected growth rate in w,

$v = \sigma_w \sqrt{w}$ = the volatility of w,

λ_w = market price of risk of w.

In pricing the WIPM contract with its default option, we make one further assumption to ensure that the model is consistent with risk-neutral world. We assume that in a risk-neutral world the housing asset will grow at the risk-free rate. To do this, we modify Equation (7) as follows,

$$\frac{\mathrm{d}H}{H} = \left(r - s\right)\mathrm{d}t + \sigma_H \mathrm{d}z_H \tag{15}$$

Using the stochastic processes specified in Equations (14) and (15), the adjusted PDE that is used in our WIPM valuation model can be written as

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[5] From Equation (10), the variable θ must satisfy $m - r = \lambda_\theta v$, so that the second term in Equation (11) becomes $\left(m - \lambda_\theta v\right)\theta \frac{\partial f}{\partial \theta} = r\theta \frac{\partial f}{\partial \theta}$.

$$\frac{1}{2}H^2\sigma_H^2\frac{\partial^2 V}{\partial H^2}+\rho H\sqrt{w}\sigma_H\sigma_w\frac{\partial^2 V}{\partial H\partial w}+\frac{1}{2}w\sigma_w^2\frac{\partial^2 V}{\partial w^2}+\left[\kappa\left(\theta_w-w\right)-\lambda_w\sigma_w\sqrt{w}\right]\frac{\partial V}{\partial w}+$$

$$\left(r-s\right)H\frac{\partial V}{\partial H}+\frac{\partial V}{\partial t}-rV=0 \qquad (16)$$

The adjusted PDE relates the WIPM value to the state variables of house price and CSW rate, and also to the risk-free interest rate. The nominal six month risk-free interest rate is determined from the classical Fisher equation of

$$r_t = {}_t\pi^{\mathrm{e}}_{t+1} + {}_t\varphi_{t+1} \qquad (17)$$

The nominal risk-free interest rate at time t for the semi-annual period $t+1$ is equal to the officially announced expected inflation rate (π^{e}) at time t for period $t+1$ plus the six monthly real rate of interest, ${}_t\varphi_{t+1}$. According to the expectations hypothesis, which assumes rational behaviour by risk-neutral investors, the yield curve will be rising if investors are expecting inflationary pressures in the future. In contrast, if the investors are anticipating disinflationary pressures, the yield curve will be downward sloping. We assume risk neutral world. Under risk neutral valuation the change in expected inflation is equal to the change in risk-free rate.[6]

Hence, solving the PDE, Equation (16), the nominal risk-free interest rate, r, is not treated as a third state variable for pricing the WIPM contract. We make the simplifying assumption that the real interest rate remains unchanged when the expected inflation increases through time. Since the expected inflation (or CSW rate) is one of the state variables of our WIPM valuation model, in the numerical solution nominal risk-free rate of interest is calculated for every possible value of CSW rate at every point in time.

Pricing the WIPM contract and default option

In order to calculate the value of the WIPM contract, it is necessary to take into consideration both the value of future monthly payments to the lender and the value of option to default on the mortgage loan. Thus, the value of the mortgage, V, is

$$V\left(H,w,t\right)=A\left(w,t\right)-D\left(H,w,t\right) \qquad (18)$$

[6] If we were valuing the mortgage in a non-risk neutral world we would need to make some arbitrary assumption about the risk premium over time.

where $A(w,t)$ represents the market value of remaining payments at time t and $D(H,w,t)$ represents the value of the default option. Since the current value of the mortgage is affected by the option to default in the future, it is necessary to use a numerical valuation procedure, which works backwards in time. Using the backward pricing method the terminal values of the future payments and the default option are known; therefore, it is possible to use Equation (16) in order to calculate the value of the mortgage in previous months.

Backward pricing methods have serious difficulties in solving the path dependency problem. In our valuation model of the WIPM contract, in order to solve the path dependency problem both with the outstanding loan balance and the valuation of default option, we basically follow Kau et al.'s (1993) methodology. However, our model is pricing a unique mortgage contract that sets its repayment schedule according to changes in CSW rate instead of a coupon rate. The basic difference that makes our problem simpler is that a typical WIPM contract does not have a contract rate, margin, or caps. Thus, instead of formulating a 'contract rate rule' as in the case of the standard ARM contract, monthly payment is directly calculated by the change in CSW rate (expected inflation) at every adjustment date. Unlike the ARM contract rate, the CSW index has no path dependency problem. The index is announced at the beginning of every January and July, and there is no mortgage payment rule that makes this announced rate depend upon the previous periods' CSW rate with the caps and floors. Thus, our valuation model does not require any auxiliary state variable to keep track of all the past values of CSW index.

The path dependency problem occurs at termination of the loan because of the unknown value of the CSW rate, which was declared at the beginning of the last semi-annual period. Since the CSW rate is one of the state variables of the valuation model, all its possible values at every time step are used along the state space. Therefore, there is no need to introduce an additional state variable to carry information about the past values of CSW rate. (The valuation of future mortgage payments on WIPM contract, the valuation of default option and solution of path dependency problem are presented in Appendix 1).

Numerical solution of PDE

Numerical solution of the main PDE, Equation (16), has an infinite domain. Therefore, the state variables of house price and wage rate should be transformed in order to eliminate their infinite boundary conditions. More specifically, it is necessary to map the infinite area $(0,\infty)\times(0,\infty)$ into the unit square $(0,1)\times(0,1)$ in order to easily solve the problem numerically. Following Azevedo-Pereira et al. (2000 and 2002) and Stanton and Wallace (1995 and 1999) the subsequent transformations are chosen for the state variables.

$$y = \frac{1}{1+\psi w}, \qquad \psi > 0 \tag{19}$$

$$x = \frac{1}{1+\omega H}, \qquad \omega > 0 \tag{20}$$

It is clear that the values used for the scale factors of ψ and ω considerably affect the density of points on the solution grid. The values for these arbitrary factors are chosen to place the values of state variables that more possibly occur in the market. We are interested in values of w in the range of 0%-2% to 15%- 20%. Therefore, $\psi = 12.5$ was chosen, which corresponds to $w = 8\%$ in the middle point of the y grid.[7]

All mortgage component values (A , D , and V) in this study are expressed as a percentage of the initial loan amount, which is set at unity. Since the initial LTV ratio is 75%, we actually use an initial house price of 1.333 TL. Therefore, a scale factor of $\omega = 0.75$ is required in the H transformation to obtain the initial value of $H = 1.333$. The main PDE is a backward parabolic equation and we transform it into a forward equation as follows (see Wilmott et al., 1993)

$$\tau = T - t \tag{21}$$

After all these transformations, the main PDE for the valuation of WIPM contract $V\left(H(x), w(y), t(\tau)\right)$ can be written as $Z(x, y, \tau)$. The transformed PDE is then given by

[7] In our comparative analysis of adjustment of short-run expected inflation (or wage rate) to long-run expected inflation, ψ = 14.2857 for w = 7% and ψ = 11.111 for w = 9% are also used.

$$\frac{1}{2}H(x)^2\sigma_H^2\omega^2x^4\frac{\partial^2 Z}{\partial x^2}+\rho H(x)\sqrt{w(y)}\sigma_H\sigma_w\psi\omega x^2y^2\frac{\partial^2 Z}{\partial x\partial y}+$$
$$\frac{1}{2}w(y)\sigma_w^2\psi^2y^4\frac{\partial^2 Z}{\partial y^2}+\left\{w(y)\sigma_w^2\psi^2y^3-\left[\kappa(\theta_w-w(y))-\sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\frac{\partial Z}{\partial y}+$$
$$\left\{H(x)^2\sigma_H^2\omega^2x^3-\left[\left(r(y)-s\right)H(x)\omega x^2\right]\right\}\frac{\partial Z}{\partial x}-\frac{\partial Z}{\partial\tau(t)}-r(y)Z=0$$

(22)

After being compacted into a unit square, the $(H\times w)$ state space is uniformly discretised as follows. For the transformed house price and wage rate variables, unit space $[0,1]$ is subdivided into I and J intervals, respectively. That is,

$$Ih=x_I=1 \quad \text{and} \quad ih=x_i \tag{23a}$$

$$Jl=y_J=1 \quad \text{and} \quad jl=y_j \tag{23b}$$

Similarly, for the time to maturity, τ, the interval $[0,T]$ is subdivided into N intervals such that

$$Nk=\tau_N=1 \quad \text{and} \quad nk=\tau_n \tag{23c}$$

Thus, the value of the mortgage $V(H,w,t)$ will be approximated by $U_{i,j}^n$. Following Kau et al. (1993) and Azevedo-Pereira et al. (2000 and 2002), a spatial grid of size 0.02 is used to discretise the (H,w) unit square. In other words, the numerical solution is obtained from 50×50 grid where $h=l=0.02$. In order to guarantee the numerical stability, 66 time steps a month is used (The convergence and stability of numerical solution of PDE are presented in Table 2 and Appendix 2).

The modified PDE in Equation (22) is approximated by the following difference equation.

$$\frac{1}{2}H(x)^2\,\sigma_H^2\omega^2x^4\,\frac{U^n_{i+1,j}-2U^n_{i,j}+U^n_{i-1,j}}{h^2}+$$

$$\rho H(x)\sqrt{w(y)}\sigma_H\sigma_w\psi\omega x^2y^2\,\frac{U^n_{i+1,j+1}-U^n_{i+1,j-1}-U^n_{i-1,j+1}+U^n_{i-1,j-1}}{4lh}+$$

$$\frac{1}{2}w(y)\,\sigma_w^2\psi^2y^4\,\frac{U^n_{i,j+1}-2U^n_{i,j}+U^n_{i,j-1}}{l^2}+$$

$$\left\{w(y)\,\sigma_w^2\psi^2y^3-\left[\kappa\left(\theta_w-w(y)\right)-\sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\frac{U^n_{i,j+1}-U^n_{i,j-1}}{2l}+$$

$$\left\{H(x)^2\,\sigma_H^2\omega^2x^3-\left[\left(r(y)-s\right)H(x)\,\omega x^2\right]\right\}\frac{U^n_{i+1,j}-U^n_{i-1,j}}{2h}-$$

$$\frac{U^{n+1}_{i,j}+U^n_{i,j}}{k}-r(y)\,U^n_{i,j}=0 \qquad (24)$$

It is important to note that the central difference approximations are used for space derivatives, while a forward difference approximation is used for the time derivative. We can arrange Equation (24) to show that the value of the mortgage at a certain time-step is a function of its own value at the previous time step. That is,

$$U^{n+1}_{i,j}=\left\{1-\left[H(x)^2\,\sigma_H^2\omega^2x^4\left(\frac{k}{h^2}\right)\right]-\left[w(y)\,\sigma_w^2\psi^2y^4\left(\frac{k}{l^2}\right)\right]-r(y)\,k\right\}U^n_{i,j}+$$

$$\left\{\frac{1}{2}H(x)^2\,\sigma_H^2\omega^2x^4\left(\frac{k}{h^2}\right)\right\}\left(U^n_{i+1,j}+U^n_{i-1,j}\right)+$$

$$\left\{\frac{1}{2}w(y)\,\sigma_w^2\psi^2y^4\left(\frac{k}{l^2}\right)\right\}\left(U^n_{i,j+1}+U^n_{i,j-1}\right)+$$

$$\left\{\left\{H(x)^2\,\sigma_H^2\omega^2x^3-\left[\left(r(y)-s\right)H(x)\,\omega x^2\right]\right\}\left(\frac{k}{2h}\right)\right\}\left(U^n_{i+1,j}-U^n_{i-1,j}\right)+$$

$$\left\{\left\{w(y)\,\sigma_w^2\psi^2y^3-\left[\kappa\left(\theta_w-w(y)\right)-\sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\left(\frac{k}{2l}\right)\right\}\left(U^n_{i,j+1}-U^n_{i,j-1}\right)+$$

$$\rho H(x)\sqrt{w(y)}\sigma_H\sigma_w\psi\omega x^2y^2\left(\frac{k}{4lh}\right)\left(U^n_{i+1,j+1}-U^n_{i+1,j-1}-U^n_{i-1,j+1}+U^n_{i-1,j-1}\right)$$

(25)

Table 1: Upwind pricing method for improving numerical stability of explicit finite difference approach

$b_1 = \frac{\partial V}{\partial H}$, the coefficient of the first derivative term with respect to H ;

$b_2 = \frac{\partial V}{\partial w}$, the coefficient of the first derivative term with respect to w .

Sign of the coefficients	Type of difference approximation
If $b_1 < 0$, $b_2 < 0$	Backward difference in H : $\frac{V_i^n - V_{i-1}^n}{\delta H}$
	Backward difference in w : $\frac{V_j^n - V_{j-1}^n}{\delta w}$
If $b_1 < 0$, $b_2 > 0$	Backward difference in H : $\frac{V_i^n - V_{i-1}^n}{\delta H}$
	Forward difference in w : $\frac{V_{j+1}^n - V_j^n}{\delta w}$
If $b_1 > 0$, $b_2 > 0$	Forward difference in H : $\frac{V_{i+1}^n - V_i^n}{\delta H}$
	Forward difference in w : $\frac{V_{j+1}^n - V_j^n}{\delta w}$
If $b_1 > 0$, $b_2 < 0$	Forward difference in H : $\frac{V_{i+1}^n - V_i^n}{\delta H}$
	Backward difference in w : $\frac{V_j^n - V_{j-1}^n}{\delta w}$

As an alternative representation of the finite difference scheme, we use the following equation, in which the approximations of the main function, $U_{i,j}^n$ s, are perfectly isolated.

$$U_{i,j}^{n+1} = \left\{1 - \left[H(x)^2 \sigma_H^2 \omega^2 x^4 \left(\frac{k}{h^2}\right)\right] - \left[w(y)\sigma_w^2 \psi^2 y^4 \left(\frac{k}{l^2}\right)\right] - r(y)k -\right.$$

$$\left\{\left\{H(x)^2\sigma_H^2\omega^2x^3-\left[(r(y)-s)H(x)\omega x^2\right]\right\}\left(\frac{k}{h}\right)\right\}-$$
$$\left\{\left\{w(y)\sigma_w^2\psi^2y^3-\left[\kappa(\theta_w-w(y))-\sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\left(\frac{k}{l}\right)\right\}\right\}U_{i,j}^n+$$
$$\left\{\frac{1}{2}H(x)^2\sigma_H^2\omega^2x^4\left(\frac{k}{h^2}\right)\right\}U_{i-1,j}^n++\left\{\frac{1}{2}w(y)\sigma_w^2\psi^2y^4\left(\frac{k}{l^2}\right)\right\}U_{i,j-1}^n$$
$$\left\{\left[\frac{1}{2}H(x)^2\sigma_H^2\omega^2x^4\left(\frac{k}{h^2}\right)\right]+\left\{H(x)^2\sigma_H^2\omega^2x^3-\left[(r(y)-s)H(x)\omega x^2\right]\right\}\left(\frac{k}{h}\right)\right\}U_{i+1,j}^n+$$
$$\left\{\left[\frac{1}{2}w(y)\sigma_w^2\psi^2y^4\left(\frac{k}{l^2}\right)\right]+\left\{\left\{\begin{array}{l}w(y)\sigma_w^2\psi^2y^3-\\ \left[\kappa(\theta_w-w(y))-\sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\end{array}\right\}\left(\frac{k}{l}\right)\right\}\right\}U_{i,j+1}^n \quad (26)$$

Obtaining Equation (26), we use one-sided difference (forward approximation) instead of a central difference for the first derivatives of the mortgage value with respect to the state variables. This is because, by using a one-sided difference, it is possible to remove the limitation on the space step size and improve stability of the numerical scheme (see Appendix 2). More specifically, in the main PDE, although the coefficients of the second derivative terms are always positive, it is not true for the coefficients of the first derivative terms. In order to keep the errors associated with the finite difference representation inside acceptable bounds, it is necessary to guarantee that all the U^n coefficients are positive. It is possible to avoid the instability problem by using forward and backward differences for the first derivative terms instead of using central differences (see Azevedo-Pereira, 1997 and Wilmott, 2000). Thus, a forward difference approach is used when the coefficient of the first derivative term is positive and a backward approach is used when it is negative. The use of one-sided differences depending on the sign of the first derivative term is called 'Upwind differencing' method (Wilmott, 2000).

Since we have two state variables in our valuation model, there are four potential combinations of first derivative signs (see Table 2). Our Matlab program code gives the appropriate form of the approximation, which is automatically calculated as a function of the sign of the first derivative in each dimension. Thus, all alternatives are considered in solving the main PDE.

Boundary conditions

The transformed structure of the state variables is used for simplifying the solution of the model; however, the corresponding state variables of H and w are still the original ones. Therefore, the mortgage component values are determined when the boundary conditions are applied at extreme house values and wage rates. More specifically, when $w = H = 0$, and when $w \to \infty$ and $H \to \infty$. It is important to note that all the formulation related to the boundary conditions of the problem considers regenerated versions of our main PDE.

When $H = 0$, the value of mortgage contract is certainly higher than the house value. Thus, the default is certain. In case of default, the value of mortgage becomes equal to the value of the house.

$$V(0, w) = H = 0 \tag{27}$$

$$D(0, w) = A(w) \tag{28}$$

When $H \to \infty$, the default does not have any value. Thus, the value of the mortgage contract is equal to the value of future payments. That is,

$$\lim_{H \to \infty} D(H, w) = 0 \tag{29}$$

$$\lim_{H \to \infty} V(H, w) = A(w) \tag{30}$$

The value of future payments, A, does not depend on H; therefore, a PDE in w alone is solved in order to obtain the value of future payments. That is,

$$\frac{1}{2} w\sigma_w^2 \frac{\partial^2 f}{\partial w^2} + \left[\kappa\left(\theta_w - w\right) - \lambda_w \sigma_w \sqrt{w}\right]\frac{\partial f}{\partial w} + \frac{\partial f}{\partial t} - rf = 0 \tag{31}$$

In terms of the explicit finite difference representation, the following equations are used in our Matlab program code.

If $\dfrac{\partial V}{\partial w} < 0$, we use the backward difference scheme as follows.

$$A_j^{n+1} = \left\{1 - \left[w(y)\sigma_w^2\psi^2 y^4\left(\frac{k}{l^2}\right)\right] - r(y)k\right\}A_i^n + \left\{\frac{1}{2}w(y)\sigma_w^2\psi^2 y^4\left(\frac{k}{l^2}\right)\right\}\left(A_{j+1}^n + A_{j-1}^n\right) +$$

$$\left\{\left\{w(y)\sigma_w^2\psi^2 y^3 - \left[\kappa\left(\theta_w - w(y)\right) - \sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\left(\frac{k}{l}\right)\right\}\left(A_j^n - A_{j-1}^n\right)$$

(32a)

If $\frac{\partial V}{\partial w} > 0$, then we use the forward difference scheme as follows.

$$A_j^{n+1} = \left\{1 - \left[w(y)\sigma_w^2\psi^2 y^4\left(\frac{k}{l^2}\right)\right] - r(y)k\right\}A_i^n + \left\{\frac{1}{2}w(y)\sigma_w^2\psi^2 y^4\left(\frac{k}{l^2}\right)\right\}\left(A_{j+1}^n + A_{j-1}^n\right) +$$

$$\left\{\left\{w(y)\sigma_w^2\psi^2 y^3 - \left[\kappa\left(\theta_w - w(y)\right) - \sigma_w\sqrt{w(y)}\lambda_w\right]\psi y^2\right\}\left(\frac{k}{l}\right)\right\}\left(A_{j+1}^n - A_j^n\right) \tag{32b}$$

When $w = 0$, there is still discounting for calculating the present market value of future payments. Because, depending on the real rate of interest nominal interest rate is not null, $r \neq 0$. It is necessary to note that a null value for the state variable does not imply a null derivative in relation to this state variable. Therefore, the following regenerated PDE and the explicit finite difference equations are used respectively in order to find the value of A when $w = 0$.

$$\kappa\theta_w \frac{\partial f}{\partial w} + \frac{\partial f}{\partial t} - rf = 0 \tag{33}$$

$$A_j^{n+1} = \left(1 - r(y)k\right)A_j^n - \left(\kappa\theta_w\psi y^2 \frac{k}{l}\right)\left(A_j^n - A_{j-1}^n\right) \tag{34}$$

Note that Equation (34) is same as Equation (32a) when we set $w(y) = 0$. The value of default, $D(H,0,t)$, and the mortgage contract $V(H,0,t)$ will be given by the solution of the regenerated form of the main PDE as follows.

$$\frac{1}{2}H^2\sigma_H^2\frac{\partial^2 V}{\partial H^2} + (r - s)H\frac{\partial V}{\partial H} + \kappa\theta_w\frac{\partial V}{\partial w} + \frac{\partial V}{\partial t} - rV = 0 \tag{35}$$

Finally, considering $w \to \infty$, and so $r \to \infty$, the assets that involve payments in the future are of no consequence. Thus, all the assets become valueless.

$$\lim_{w\to\infty} A(w) = 0 \tag{36}$$

$$\lim_{w\to\infty} D(H,w) = 0 \tag{37}$$

$$\lim_{w\to\infty} V(H,w) = 0 \tag{38}$$

The market price of expected inflation (CSW rate) risk

The standard capital asset pricing model allows us to determine the market price of expected inflation risk (or the market price of risk of w) when historical data are available for w (see Dixit and Pindyck, 1994 and Hull, 2003). Using the base parameter values of mean, θ_w, standard deviation, σ_w, and speed of mean reversion, κ, we can estimate the market price of expected inflation risk, λ_w, for any possible wage rate, w, and the corresponding risk-free rate of interest, r, as follows

$$\lambda_w = \frac{\kappa(\theta_w - w) - r}{\sigma_w \sqrt{w}} \tag{39}$$

The value of λ_w gives the expected inflation risk premium; that is the basic relationship between the risk-free interest rate and the expected growth rate of w. A positive value of λ_w implies that the expected growth rate of wage, g_w, is higher than the nominal interest rate, r. On the other hand, a negative value of λ_w implies that nominal interest rate is higher than the expected growth rate of wage. Thus,

$$\text{If} \quad \lambda_w > 0, \qquad \kappa\left(\theta_w - w\right) = g_w > r \tag{40a}$$

$$\text{If} \quad \lambda_w < 0, \qquad \kappa\left(\theta_w - w\right) = g_w < r \tag{40b}$$

In order to capture the effect of term structure of expected inflation rate, we can set the short-term wage rate at higher or lower levels and analyse the value of the mortgage contract for the different measures of λ_w. It is important to note that following Wilmott (2000) we allow λ_w to depend on wage rate but not on time.[8] In other words, we make the simplifying assumption that the real rate of interest is constant over the mortgage contract term.

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[8] Assuming the spot interest rate follows the stochastic process of $d_r = u\left(r\right)\mathrm{d}t + wr^{\beta}\mathrm{d}X$ (where β=0 for Vasicek and Ho &Lee, and β=1/2 for CIR model), Wilmott (2000) analyses the relationship between the slope of the yield curve and the market price of risk. He allows the market price of risk of r, λ_r to have only a spot-rate dependence being independent of time.

Numerical Results

Because the WIPM is a recently originated contract there is limited historical information available on this mortgage contract. For the simulation of the contract we require the base case parameters for the two state variables, namely the house price and the CSW rate (expected inflation). In line with some of the existing studies in the literature we calculate the historical volatility of house price using the monthly Housing Price Index published by the State Institute of Statistics between February 1994 and February 2004. The expected inflation rate (CSW rate), is announced semi-annually by the Ministry of Finance. The semi-annual observations on the expected inflation are available only for the period from January 1993 to January 2004. Using this limited sample of 23 semi-annual observations, we estimate the long-term mean and volatility of expected inflation. In order to ascertain the reliability of these estimates, we compare them against the corresponding historical mean and volatility of the actual inflation over the longer sample period from February 1987 and February 2004.[9]

The historical average and volatility estimates of the expected inflation are 25.5% and 11% respectively. The historical average and volatility estimates of the actual inflation are 33% and 15.3%, respectively. Both the average and volatility of the actual inflation, over the sample period 1987 to 2004, are higher than those of the expected inflation over the period 1993 to 2004. Given that the estimates of the expected inflation are lower than the actual inflation, the value of the WIPM contract in the base case scenario would be expected to be lower than otherwise. The semi-annual CSW rate announced by the Ministry of Finance in January 2004 is 8%. For the base case scenario simulation of the WIPM contract we set the initial CSW rate to be 8%. Since this initial value is well below its long-term historical average of 25.5%, as well as 33% actual inflation rate, the short-run CSW rate is expected to adjust towards its long-term average. In 2002 and 2003 the inflation dropped to the lowest level over the past sixteen years, resulting in low values of expected inflation (CSW rates) in the recent years. The long-term semi-annual average rate of 25.5% is because of the relatively higher expected inflation observed between 1995 and 1999.

As explained earlier, the real interest rate in Turkey has fluctuated

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[9] In the US literature, in pricing FRMs and ARMs, it is a standard practice to set the base parameter values of interest rate by using the parameter estimates from Buser and Hendershott *et al.* (1984) and Titman and Torous (1989). Titman and Torous (1989) stated that, while it is possible to identify interest rate volatility, it is not possible to identify separately the parameters of long-term mean, θ_r, and speed of adjustment, κ. Thus, without loss of generality, the authors specify these parameters consistent with observed government bond prices (see Titman and Torous, 1989).

considerably over the last decade and has been negative in some periods, especially between 1987 and 1997. Table 1 shows that, between 1987 and 1997, prior to the origination of the WIPM contract, the average annual real interest rate is –4.4% (compound average) and –7.4% (simple average). For the whole period, between 1987 and 2003, the average annual real rate of interest is 0.85% (compound average) and 1.2% (simple average). It is only between 1997 and 2003 that real rate is observed to be 7.6% (compound average) or 9.6% (simple average). On average, the annual real interest rate has predominantly been negative over the past decade. In our base case scenario we have therefore decided to use –2% real interest rate, which is the best representation of the economic environment between 1987 and 1997. For our alternative scenarios, we use zero and positive real interest rate in order to analyse the sensitivity of the value of the WIPM contract for the lender under different economic environments.

Table 2: Average values of annual real interest rates

Time period	Real $r = r - \pi^a$	Real $r = \dfrac{(1+r)}{(1+\pi^a)} - 1$
1987-1997	–7.40 %	–4.40 %
1997-2003	9.60 %	7.60 %
1987-2003	0.85 %	1.20 %

Simulation results of the base case scenario

This section aims to illustrate that backward pricing method can be applied for pricing the index-linked mortgages. The values of remaining mortgage payments, A, default option, D, and the entire contract value, V, at the origination of the mortgage are presented in Figures 4 to 6, respectively. The values of A, D, and V, are expressed as the par value of the loan; that is, as a percentage of the loan amount which is set to unity. The main reason for presenting these figures is to demonstrate both the smoothness of the numerical solution over the grid and economic consistency of the simulation results.

The value of future payments, A, depends only on wage rate, w, and not on house prices, H. The initial wage rate (w) in January 2004 is 8%, which is well below the long-term mean (θ_w) of 25.5%. Since this initial value is relatively low, we expect it to converge towards its long-term mean over the life of the mortgage contract. When w is closer to θ_w, the expected growth rate of w, that is $\kappa(\theta_w - w)$, would be lower than the nominal risk-free interest rate (r). Figure 4 shows the book value and market value of future WIPM payments, respectively, over an appropriate range of wage rates. Figure 4a shows that the book value of promised mortgage payments increase with the wage rate. In contrast, the present value of future WIPM payments declines with w . This is because the nominal risk-free interest rate, r , which moves in line with the expected inflation (or the wage rate), also rises. Thus, a lower expected growth rate in w and the correspondingly higher value of the discount factor decrease the value of A (see Figure 4b).

Figure 4a: Book value of future payments on WIPM contract over an appropriate range of wage rates

Figure 4b: Market value of future payments on WIPM contract (*A*) over an appropriate range of wage rates

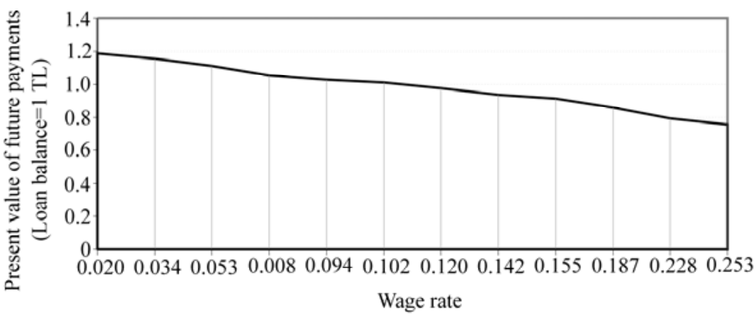
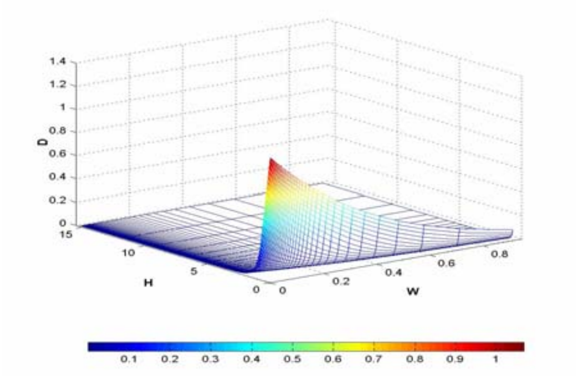


Figure 5 shows the value of the default option, D . The relationship between the house price, H , and the value of the mortgage contract is the greatest influence on the value of default option. The value of D is positive in almost all of the subset of the state space where $H < H(0)$. As the increase in the level of w leads to decreases in the value of A and V , the value of default option, whenever positive, tends to be inversely related to w . The combined effect of low house price with low level of wage rate result in extremely high default value as seen from Figure 5.

Figure 5: The value of default option *(D)*



The value of the mortgage contract, V , is a function of both A and D . A low level of house price tends to increase the value of the default option, D , held by the borrower and consequently to reduce the value of the mortgage contract. As Figure 6a shows, at a low level of house price, the value of V decreases sharply and reaches to a very low level. Default is certain to occur at the next payment date when house price is at a very low level; thus, the level of the house price exerts the greatest influence on the value of the mortgage contract. The evolution of wage rate has a direct effect on the value of payments A ; therefore, the relationship between the evolution of the wage rate and the value of the mortgage contract operates through the effect caused for this evolution in terms of A . At higher levels of house prices, the value of the mortgage contract has an inverse relationship with the changes in wage rate. The numerical results show that increase in w leads to decrease in A and also V .

In order to examine the book value of WIPM contract we solve the same PDE, but with different terminal conditions. For the purposes of internal valuation, the Emlak Bank uses the book value of the WIPM contract rather than the market value. Thus, we calculate the book value of the WIPM repayments without discounting the payments at the market interest rate. Figure 6b shows the book value of the WIPM contract at the mortgage origination. The effect of changes in house prices on the WIPM contract is straightforward. As in the case of market value of the contract (Figure 6a), low levels of house prices tend to reduce the book value, while, at high levels of house prices, it moves in the same direction as w . This is because when wage rate increases the nominal value of future payments also increases and results in higher book value of the contract.

Figure 6a: The value of mortgage contract *(V)*

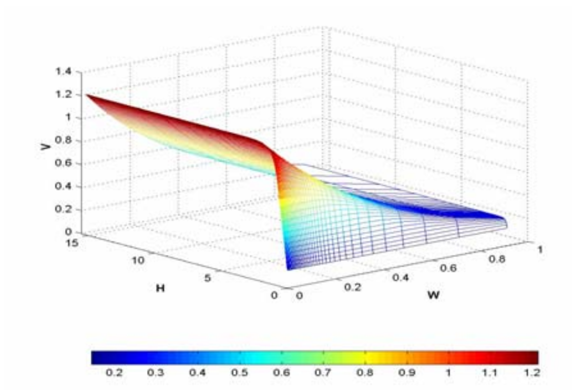
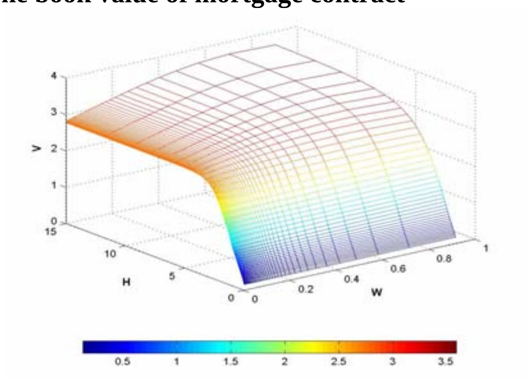


Figure 6b: The book value of mortgage contract



Default option value and the WIPM contract value at origination of the mortgage under the base case scenario. The calculations that underlie this graph are done using the base parameter values: initial wage rate, $w(0)$, 8%; initial risk-free interest rate, $r(0)$, 6%; long-term mean of wage rate, θ_w, 25.5%; wage rate volatility, σ_w, 15%; house price volatility, σ_H, 10%; 75% LTV ratio; market price of expected inflation risk, λ_w, 3%; mean reversion rate, κ, 35%; house service flow, s, 6.25%; and a correlation coefficient, ρ, 60%.

Effect of changes in the economic environment

Wage rates and their volatility

This section analyses the partial effects of changes both in the initial wage rate and the volatility of wage rate. By changing the initial level assumed by the wage rate $w(0)$, while holding its long-term mean value θ_w constant for all runs, it is possible to analyse the effect of different shapes of the convergence path of the expected wage rate.[10] In other words, we can perform a comparative analysis of the adjustment of short-run expected inflation to the long-run expected inflation.

As Table 3 shows the market value of future mortgage payments, A, falls when the initial wage rate, $w(0)$, is high. This is because when $w(0)$ is at a relatively higher level, its convergence path to the long-run trend is flatter. Thus, as the expected growth in w declines, the market value of the promised mortgage payments fall. After setting the initial wage rate at 8%, as the base case, we take the range 7% to 9% as the high and low values. Changing the initial wage rate while holding its volatility, σ_w, constant gives different measures for the market price of expected inflation risk as given by Equation (15).[11]

For the base value of 8%, the market price of expected inflation risk is calculated as $\lambda_w = 0.03$. When $w(0)$ is set at 7% the market price of expected inflation risk becomes significantly higher, that is $\lambda_w = 0.37$, due to the higher excess inflation risk premium. In contrast, setting $w(0)$ at 9% results in negative market price of expected inflation risk, that is $\lambda_w = -0.27$, because of the lower growth rate in the expected inflation in comparison to risk-free market interest rate (see Table 4). λ_w measures the risk return trade-off for securities dependent on the wage rate. For high values of $w(0)$, the expected inflation risk premium declines, leading to lower and even

[10] The convergence path here represents the expected growth rate of CSW rate, $g_w = \kappa(\theta_w - w)$ as defined in Equation 40.

[11] As we move from one market price of risk to another, the expected growth rates of security prices change, but their volatilities remain the same. Choosing a particular market price of risk is also referred to as defining the probability measure (see Hull, 2003).

negative values of λ_w. Therefore, the market value of A, that is dependent on w through $A(w,t)$, also decreases.

Table 3: Combined effects of changes in LTV ratio and the initial house price *H*(0) and initial wage rate *w*(0)

(All results to Par value of a 10-year mortgage loan, in %)

H(0)	**Future payments (*A*)**			**Default (*D*)**			**Mortgage (*V*)**		
	w(0)= 7%	*w*(0)= 8%	*w*(0)= 9%	*w*(0)= 7%	*w*(0)= 8%	*w*(0)= 9%	*w*(0)= 7%	*w*(0)= 8%	*w*(0)= 9%
65% LTV ratio									
20% drop in house value	121.36	105.62	91.14	9.15	4.51	1.94	112.21	101.11	89.20
10% drop in house value	121.36	105.62	91.14	6.96	3.32	1.38	114.40	102.30	89.76
Original house value	121.36	105.62	91.14	5.20	2.39	0.96	116.16	103.23	90.18
10% increase in house value	121.36	105.62	91.14	3.79	1.69	0.66	117.57	103.93	90.48
20% increase in house value	121.36	105.62	91.14	2.70	1.16	0.44	118.66	104.46	90.70
75% LTV ratio									
20% drop in house value	121.36	105.62	91.14	17.35	9.17	4.23	104.01	96.45	86.91
10% drop in house value	121.36	105.62	91.14	13.64	6.96	3.10	107.72	98.66	88.04
Original house value	121.36	105.62	91.14	10.57	5.21	2.24	110.79	100.41	88.90
10% increase in house value	121.36	105.62	91.14	8.04	3.83	1.59	113.32	101.79	89.55
20% increase in house value	121.36	105.62	91.14	6.00	2.76	1.11	115.36	102.86	90.03

Table 4: Combined effects of the market price of expected inflation (CSW rate) risk, λ_w and CSW rate volatility, σ_w

w(0)	λ_w	**Future payments (*A*)**			**Default (*D*)**			**Mortgage (*V*)**		
		σ_w= 10%	σ_w= 15%	σ_w= 20%	σ_w= 10%	σ_w= 15%	σ_w= 20%	σ_w= 10%	σ_w= 15%	σ_w= 20%
7%	0.37	114.35	121.36	128.50	5.90	10.57	16.13	108.45	110.79	112.37
8%	0.03	103.12	105.62	108.61	3.07	5.21	7.90	100.05	100.41	100.71
9%	–0.27	92.61	91.14	90.46	3.17	2.24	1.81	89.44	88.90	87.20

By changing the initial level assumed by the wage rate w(0), while holding constant its long-term mean value θ_w for all runs, it is possible to calculate different market price of expected inflation risk, λ_w. Market price of expected inflation risk measures the risk return trade-off for securities dependent on the expected inflation (or wage rate).

The default option, D, component of the WIPM contract is directly affected by the market value of future mortgage payments, A, because its payoff is calculated as $A(w,t)-H(t)$. Thus, the default option is inversely related to $w(0)$. The adverse effect of an increase in $w(0)$ on the value of future payments is not offset by the value of option to terminate the loan by default. Thus, the overall effect of an increase in wage rate on the value of the entire contract, V, is negative (see Table 4).

Examining the effect of wage rate volatility, σ_w, our numerical results show that the market value of the future WIPM payments increase with increase in volatility when the market price of expected inflation risk, λ_w, is positive (see Table 4). This is because when λ_w is positive, an increase in wage rate volatility results in a higher expected rate of return on w, that is a higher g_w. Thus, the market value of future payments on WIPM increases with the wage rate volatility due to the higher excess return on w (greater expected inflation risk premium) over the risk-free rate of interest, r.

$$g_w = r + \lambda_w \sigma_w \sqrt{w} \tag{41}$$

It is important to note that the increase in market value of future payments is more significant when the market price of expected inflation risk is higher, $\lambda_w = 0.37$, at 7% initial wage rate. On the other hand, as would be expected when λ_w is negative at 9% initial wage rate, the market value of future payments falls as wage rate volatility rises.

House prices and their volatility

The partial effects induced by changes in the initial house price, $H(0)$, and house price volatility, σ_H, are presented in Tables 3 and 5 respectively. The effect of house price volatility, σ_H, on a WIPM is straightforward. The numerical results confirm the general expectations that the value of future payments, A, remains unchanged with an increase in σ_H. This is because the value of A is affected by neither house price nor its volatility. However, the value of the default option, D, increases with house price volatility, σ_H, and results in lower values of the entire mortgage contract, V.

Table 5: Combined effects of changes in LTV ratio, house price volatility (σ_H) and wage rate volatility (σ_w)

LTV ratio	σ_w	Future payments (A) σ_H =5%	σ_H =10%	σ_H =15%	Default (D) σ_H =5%	σ_H =10%	σ_H =15%	Mortgage (V) σ_H =5%	σ_H =10%	σ_H =15%
65%	10%	103.12	103.12	-	0.28	1.20	-	102.84	101.92	-
	15%	105.62	105.62	105.62	0.88	2.39	4.53	104.74	103.23	101.09
	20%	-	108.61	108.61	-	4.05	6.63	-	104.56	101.98
75%	10%	103.12	103.12	-	1.20	3.07	-	101.92	100.05	-
	15%	105.62	105.62	105.62	2.71	5.21	8.21	102.91	100.41	97.41
	20%	-	108.61	108.61	-	7.90	11.04	-	100.71	97.57

The effect of changes in loan-to-value (LTV) ratio on the value of default option is also displayed in these tables. It can be easily observed from Table 3 that a fall in the original house price (just after the loan origination) can substantially raise the value of default, D , particularly at a higher LTV ratio. For a given house price, a rise in the LTV ratio corresponds to a higher loan amount, resulting in a naturally growing A . Therefore, the one-to-one relationship between the amount of the loan and the original value of the house implies that the probability of default rises. A decrease in the original value of the house increases the default value even further, since in these circumstances it is likely that the outstanding debt surpasses the value of the house.

The maximum initial LTV ratio for the WIPM contract is 75%; hence we carried out simulations with an alternative LTV ratio of 65%. The value of A remains unchanged with higher LTV ratio since these values are being expressed as par. More specifically, the value of A and D for the LTV ratio of 65% are scaled values calculated by re-setting the initial loan amount to unity and changing the original house value correspondingly.[12] Unlike the effects of σ_H , the value of entire contract, V , is positively related to house price, H . Since the default option value, D , is inversely related to H , the value of the mortgage contract, V , rises with H . The combined effects of the variations in house price and wage rate volatilities at each level of LTV ratio are also given in Table 5. According to our numerical results, house

[12] For the original LTV ratio of 75% with the base parameter values, the simulations are carried out with an initial loan amount that is set at unity, L=1, and mortgage component values are calculated for the original house price of H=1.333. For the LTV ratio of 65%, the simulations are carried out with the initial loan amount of L=0.8666, when H=1.333. The scaled values of A, which correspond to unity loan amount, are obtained by dividing A to 0.8666 since the value of promised payments is homogenous of degree one in the current loan amount.

price volatility, σ_H, has a greater impact on the default option value compared to the wage rate volatility, σ_w, at each level of the LTV ratio.

The value of lender's position

In an equilibrium framework the terms of any contract cannot be set arbitrarily. A mortgage contract can only be acceptable if it represents a fair deal for both parties to be able to accept the contract agreement. More specifically, it is necessary to guarantee that the borrower is not able to make an instantaneous profit by terminating the loan at origination and, similarly that the contract is not structured in such a way that allows the lender to make any immediate profit. This is a condition of no arbitrage. The arbitrage principle requires that the assets or claims exchanged be of equal value for both parties to accept the transaction. Therefore, for the WIPM contract the following condition must be satisfied at the origination of the contract

$$V_L\left(H(0), w(0), 0\right) = V_B\left(H(0), w(0), 0\right) = V \qquad (42)$$

The value of WIPM contract to the lender is simply equal to the value of mortgage to the borrower. The absence of a coupon rate makes it impossible for the WIPM design to find a contract rate capable of generating fair deals for lenders and borrowers. More specifically, the equilibrium condition for a standard non-insured, defaultable mortgage loan, $V_B\left(c\right) - L = 0$, does not exist for a WIPM. Thus, the lender's position in a WIPM contract is

$$V\left(H(0), w(0), 0\right) - L \qquad (43)$$

where L is the loan amount. We find that the value of the contract to the lender is significantly higher when market price of expected inflation risk is positive. This implies that mortgage lender benefits from the excess return on expected inflation risk. It is possible to say that it is the positive unexpected inflation that decreases the real rate of interest, resulting in excess return on the expected inflation. LTV ratio is an important factor in the determination of the contract value to the lender. At the initial house value, originating WIPMs with a lower LTV ratio of 65% results in considerably higher value of the lender position. The combined effect of a lower LTV ratio and a higher initial house value (just after the loan origination) provides considerable benefits for the lender (See Table 3). It is important to note that a high volatility of house price, which greatly affects the value of the default option in comparison to the wage rate volatility, significantly reduces the value of lender's position.

For standard mortgage contracts, lenders can manipulate the combinations of contractual features (mortgage insurance, arrangement fee, prepayment penalty, etc.) in order to reach different contract rates for loans that are equally attractive in economic terms. However, Emlak Bank does not rely on insurance market to share the default risk attached to WIPM loans. It would be advantageous for the lender to include the mortgage insurance in the contract design of the WIPM; thereby the value of lender's position can become positive for lower and even negative values of expected inflation risk premium.

We also analyse the value of lender's position under alternative base case scenarios. The previous numerical results are based upon the base case parameter values, which are determined by taking into consideration the historical estimates of actual inflation. Now we will use a wider range of parameter values are used in order to analyse the sensitivity of the value of the WIPM contract for the lender under different economic conditions.

Table 6 demonstrates the conditions under which the lender benefits from originating WIPM. Given that the historical average estimate of the actual inflation (33%) is higher than that of the expected inflation (25.5%), we set the long-term mean, θ_w, to be 30% as an alternative base case scenario. The CSW rate is declared to be 6% for the second half of 2004. Therefore, we set the alternative initial CSW rates between 5% and 10%. As alternative to –0.02 real rate of interest, which represents the economic environment between 1987 and 1997, we use zero and 0.02 real interest rates.

Table 6: Value of the WIPM contract to the lender under alternative base case scenarios

σ_w = 10%

	θ_w = 25.5%			θ_w = 30%		
	w_0 = 5%	w_0 = 8%	w_0 = 10%	w_0 = 5%	w_0 = 8%	w_0 = 10%
Real r = –0.02	130.82	103.26	76.01	131.22	107.94	80.13
Real r = 0.00	117.92	76.98	55.09	121.81	83.16	63.18
Real r = 0.02	98.09	61.57	47.37	105.27	69.04	51.47

σ_w = 15%

	θ_w = 25.5%			θ_w = 30%		
	w_0 = 5%	w_0 = 8%	w_0 = 10%	w_0 = 5%	w_0 = 8%	w_0 = 10%
Real r = –0.02	129.89	100.41	78.90	130.52	106.84	81.56
Real r = 0.00	116.38	79.58	57.62	120.08	84.10	65.87
Real r = 0.02	97.41	62.05	48.60	103.47	70.18	53.06

A wide range of parameter values are used in order to analyse the sensitivity of the value of the WIPM contract for the lender under different economic conditions.

As seen from Table 6, higher long-term mean of 30% increases the expected growth rate of wage rate and results in considerably higher values of the

WIPM contract for the lender. At a higher initial wage rate of 10% it is not rational for the lender to originate the WIPM. This is firstly because relatively lower expected growth rate of wage rate (or expected inflation) results in negative expected inflation risk premium for the lender. Second, the correspondingly higher interest rate (discount factor) results in lower WIPM value. In contrast, the lender has positive inflation risk premium and higher values of the contract, for each scenario of real interest rate, when the initial wage rate is 5%. Therefore, we argue that as the wage rate becomes closer to the long-term mean value, reflecting a stable economic environment, WIPM is not an appropriate mortgage design for the lender. If stable economic conditions can be achieved for a certain period of time, it would certainly be better for the lender either withdraw the WIPM instrument or convert it into the standard mortgage instruments of moderate inflation environments such as ARMs.

The WIPM contract design works properly during the periods when the real rate of interest is negative. Therefore, the lender can greatly benefit from originating inflation-linked mortgages like WIPM when the real interest rate is negative. It would be irrational however for the lender to originate WIPMs when real rate of interest is highly positive.

Concluding Remarks

In high inflation economies inflation-indexed mortgage contracts have been widely originated. Using the standard contingent claims approach, this paper develops a valuation model for pricing a specific inflation-indexed mortgage instrument that was extensively originated in Turkey during the late 1990s inflationary period. The Turkish government in cooperation with a state-owned bank created this specific mortgage design for middle-income civil servants, who are the main group of borrowers of housing loans with their state-guaranteed salaries. Increasing the public sector wage rate in line with the expected inflation, the government aimed at facilitating both the long-term mortgage lending and borrowing.

Money market rate in Turkey has been highly volatile over the last fifteen years. During the periods of financial crises in 1994 and 2001 the real interest rate has soared to extremely high levels. In contrast, between 1987 and 1997 (excluding 1994 financial crisis), the average annual real interest rate has predominantly been negative. Analysing the Turkish treasury interest rate behaviour, recent empirical evidence indicates that the nominal interest rate increases less than the expected inflation, resulting in declining real rate of interest, when the inflation risk is positive. Under these circumstances, the lender benefits from originating mortgages linked to the expected inflation rather than highly volatile market interest rate. Thus, in an environment with uncertain inflation, a nominal mortgage contract is an extremely risky asset for the mortgage lender. Its real capital value is highly sensitive to inflation. In particular, when there are positive shocks to inflation, real rate of interest becomes negative due to higher inflation. The lender therefore has negative rate of return from originating the standard mortgage contracts.

Using the explicit finite difference methodology, we evaluate the expected inflation-indexed mortgage (WIPM) contract with its embedded default option from the lender's perspective. Our numerical results show that positive expected inflation risk premium notably increases the value of future payments on WIPM contract, resulting in high values of lender's position in the mortgage agreement. The value of promised payments increases further with a rise in wage rate volatility. This implies that mortgage lender greatly benefits from the excess return on the expected inflation (wage rate) over the risk-free interest rate. The numerical results also show that the house price volatility has a greater impact on default values in comparison to the wage rate volatility; therefore, significantly decreases the value of lender's position in the deal.

The most notable result found in this study is that inflation uncertainty should be taken into account when designing mortgage contracts in inflationary environments. In a high inflation economy like Turkey, the real interest rate decreases and even becomes negative with positive unexpected inflation. Under these circumstances, the WIPM contract provides a profitable asset to the lender due to the decreasing value of the opportunity cost of capital. In contrast, it would be irrational for the lender to originate WIPM when real rate of interest is highly positive. This study leads us to conclude that the WIPM contract design is an innovative mortgage instrument. Given that this contract combines two economic policy targets, namely facilitate housing finance for an expanding population and create mortgage market under highly volatile inflationary environment, we believe that policy makers in other emerging economies may also adopt this mortgage instrument.

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Appendix 1: Modelling Monthly Payments on WIPM

To describe the process of semi-annually adjusted mortgage repayments, the following notation is used (see Erol and Patel, 2004, 2005):

L = The total loan amount;
n = The mortgage contract maturity in terms of months;
i = The i-th adjustment period for mortgage payment;
The total number of adjustment periods =n/ reset frequency.
Thus, $i = 1, 2, ...I$ where $I = 120/6 = 20$
j = The j-th monthly payment date in the i-th adjustment period, where $0 \leq j \leq 6$
$OB(i, j)$ = Outstanding balance after the payment at (i, j);
$w(i,0)$ = Semi-annual cumulative increase in CSW index (the expected inflation) for the i-th adjustment period;
$MP(i)$ = The monthly payment of the mortgage at i-th adjustment period;
η_i = The number of remaining months from adjustment period's beginning to the contract maturity for $i = 1$ $\eta_i = n$, and for $i = 2$ to 20 $\eta_i = \eta_{i-1} - 6$

In a WIPM contract, the first period of the contract (for the first six months) mortgage repayment schedule is a fixed amount, which is calculated by dividing the total loan amount by the mortgage term, and there after at each monthly payment date, the outstanding balance of the borrower's debt decreases by the fixed amount of MP.

$$\text{MP}(1) = \frac{L}{n} \tag{A.1}$$

$$\text{OB}(1, j) = \left[L - \left(\text{MP}(1) * j\right)\right] \qquad \text{for } j = 0,...,6 \tag{A.2}$$

From the beginning of the second period, mortgage repayment schedule behaves as an adjusted payment mortgage, and the outstanding balance is adjusted semi-annually in line with change in CSW rate. Monthly payments are calculated as

$$\text{MP}(i) = \left[\text{OB}(i,0) * \frac{\left(1 + w_{(i,0)}\right)}{\eta_i}\right] \qquad \text{for } i = 2,3,\ldots, I \tag{A.3}$$

Thus, $\text{OB}(i,0) * \left(1 + w_{(i,0)}\right)$ is the CSW rate-adjusted outstanding balance that determines the monthly payments with time to maturity parameter, and $\mathrm{OB}(i,0) = \mathrm{OB}(i-1,6)$ implies that the remaining OB at the end of the 6th month of adjustment period $i-1$ equals to the outstanding balance at the beginning of period i. The outstanding debt amount after the payment date $t(i,j)$ is

$$\mathrm{OB}(i,j) = \left[\mathrm{OB}(i,0)\left(1+w_{(i,0)}\right)\right] - \left(\mathrm{MP}(i) * j\right) \quad \text{(A.4)}$$

And note that for $i = 2$, $\mathrm{OB}(2,0) = L - \left(\mathrm{MP}(1) * 6\right)$.

The promised mortgage payments

The valuation process proceeds from the maturity of the mortgage backwards in time; therefore, as the terminal condition, the value of remaining payment should be equal to the final monthly payment:[13]

$$A^{-}\left[w, t\left(I,6\right), \mathrm{OB}\left(I,0\right)\right] = \mathrm{PVMP}_{I,6}\left[\mathrm{OB}\left(I,0\right)\right] \quad \text{(A.5)}$$

The path-dependency problem occurs because the outstanding balance at the last adjustment period's beginning $\mathrm{OB}\left(I,0\right)$ is not known. It depends on all previous unpaid balances. Observing that the value of promised payments is homogeneous of degree one in the current remaining principal amount can solve this path-dependency problem, $\mathrm{OB}\left(i,j\right)$. Following the basic methodology used in Kau et al.'s (1990, 1993) and Stanton and Wallace's (1995,1999) studies, we regularly set the current unpaid balance to unity (1 TL) and then rescale it as required. We calculate MP_i when $\mathrm{OB}\left(i,0\right) = 1$, then moving backward in time we adjust $A\left[t\left(i+1,0\right)\right]$ or $A\left[t\left(i,6\right)\right]$ value to the remaining OB value. In other words, when we want to change this unpaid principal by some proportion, we need to change the value of A in that same proportion. This allows us to delete $\mathrm{OB}\left(i,j\right)$ notationally, whenever its value is taken to be unity.

Given the 1 TL outstanding balance $\mathrm{OB}\left(I,0\right)$ and the assumed six-month cumulative change in CSW index at the beginning of the last adjustment period $w\left(I,0\right)$, then MP_I is determined by Equation (A.3) where $i = I$. Present value of the last monthly payment, $\mathrm{PVMP}_{I,6}$ is determined as the last

[13] A distinction will be made between the values of an asset immediately before and immediately after the payment is made, so the notation of F^{-} (.) and F^{+} (.) will be used respectively.

term in the following equation:

$$\text{PVCF}_{20} = \frac{\text{MP}_{115}}{\left(1+\frac{r}{6}\right)} + \frac{\text{MP}_{116}}{\left(1+\frac{r}{6}\right)^2} + \frac{\text{MP}_{117}}{\left(1+\frac{r}{6}\right)^3} + \ldots + \frac{\text{MP}_{120}}{\left(1+\frac{r}{6}\right)^6} \qquad \text{(A.6)}$$

Equation (A.6) gives the present value of the expected repayments over the final six months by discounting the monthly payments using the nominal six-month risk-free interest rate. This is the present market value at the beginning of the final period, at time $t = 115$.[14]

Now, the PDE can be solved backward using terminal condition, Equation (A.5), until the beginning of the month, when a payment is due. In general, if we are at such a payment date $t(i, j)$, $j \neq 0$ or 6, then we have solved for $A^{+}\left[w, t(i,0)\right]$ and we may then write the terminal condition between months of adjustment period as

$$A^{-}\left[w, t(i,j)\right] = A^{+}\left[w, t(i,j)\right] + \text{PVMP}_{j} \quad \text{for } j = 1 \text{ to } 5 \qquad \text{(A.7)}$$

That is, moving backward in time, as each monthly payment date is reached, the value of promised payments, A, changes by the amount of the monthly payment. At the beginning of the adjustment period, time $t(i,0) = t(i-1,6)$, the subsequent outstanding balance $\text{OB}(i-1,0)$ must be set to unity. The assumed wage rate, $w(i-1,0)$ and the requirement that $\text{OB}(i-1,0) = 1$ determine MP_{i-1}. We can now calculate PVCF_{i-1} and obtain the value of $\text{OB}(i,0)$ through the equation of

$$\text{OB}(i,0) = \text{OB}(i-1,0) - \text{PVCF}_{i-1} \qquad \text{(A.8)}$$

These together give the terminal condition on adjustment dates:

$$A^{-}\left[w, t(i-1,6)\right] = A^{+}\left[w, t(i-1,6)\right]\text{OB}(i,0) + \text{PVMP}_{i-1,6},$$

[14] The basic rule in determining the monthly payments on a WIPM contract is totally different from the ARM by not taking into consideration the present value of the expected cash flows and equating them to the loan balance in order to achieve a fair mortgage pricing. Although the repayments on WIPM are adjusted to expected inflation, and so reflect the fluctuations in the economy, the bank does not discount the expected future mortgage payments by the rate of return offered by comparable investment alternatives. In order to calculate the present value of all promised payments on a WIPM, we describe present value of cash flows (PVCF) for each adjustment period throughout the mortgage term. A detailed discussion of monthly repayment and outstanding loan balance valuation can be obtained from the authors.

$$t(i-1,6) = t(i,0) \tag{A.9}$$

Thus, the value of $A^{-}\left[w,t(i,0)\right]$ is adjusted to correspond to the value at which the outstanding balance $\mathrm{OB}(i,0)$ has been reset. At this point, it is crucial to note that by adjusting the value of future payments at time $t(i,0)$ to the remaining loan balance at time $t(i-1,6)$, which gives the present value of $\mathrm{OB}(i,0)$ at time $t(i-1,0)$, it is possible to obtain the present value of all past months at the beginning of the period $i-1$.

The default option and solution of path dependency problem

The value of the option to default depends on both the house price and CSW index. If the house price is different from the value of the remaining payments, the financially rational borrower either does nothing, or sells, or defaults and gives up the house to the lender if that proves to be the most advantageous solution from a financial point of view. At expiry of the mortgage, the borrower holds the house and has an obligation to make the last mortgage payment but she also has a put option on the house $D(H,w,t)$ allowing her to default and give up the house if she wishes. Therefore, the position of the borrower at maturity is, $H + D(H,w,t) - \mathrm{MP}_I$ with the following mortgage value:

$$V^{-}\left[H,w,t(I,6)\right] = \min\left[\mathrm{PVMP}_{I,6}, H\right] \tag{A.10}$$

Similarly, at any payment date $t(i,j)$ other than the adjustment date the borrower holds a corresponding position $H + D(H,w,t) - A^{-}(w,t)$ with the mortgage value of

$$V^{-}\left[H,w,t(i,j)\right] = \min\left[V^{+}\left(H,w,t(i,j)\right) + \mathrm{PVMP}_j, H\right] \quad \text{for } j = 1 \text{ to } 5 \tag{A.11}$$

The only complication arises when the beginning of any adjustment date $t(i,0)$ is reached. As in the case of promised payments, the previous period's outstanding balance $\mathrm{OB}(i-1,0)$ is to be set to unity. However, unlike the payments $A(w,t)$ default is not homogeneous in OB because it also depends on the house price H. That is, doubling the outstanding balance doubles the value of payments, but it does not double the value of default on a house by itself. However, while default is not homogeneous in OB itself, it is homogeneous both in H and OB together. It means that the value of default becomes twice as great when the loan becomes twice the amount and the house becomes twice as valuable. Thus, at the beginning of adjustment periods the mortgage value is (see Kau et al., 1993)

$$V^{-}\left[H, w, t(i-1,6)\right] = \min\left\{V^{+}\left[\frac{H}{OB(i,0)}, w, t(i-1,6)\right]\mathrm{OB}(i,0) + \mathrm{PVMP}_{i-1,6}, H\right\} \tag{A.12}$$

where both $\mathrm{OB}(i,0)$ and $\mathrm{PVMP}_{i-1,6}$ are determined by the assumed wage rate $w(i-1,0)$ and $\mathrm{OB}(i-1,0)=1$. The default decision is assumed not to be simply triggered whenever the present value of the remaining payments exceeds the current market value of the house H but rather whenever V the value of the mortgage to the borrower exceeds the house value.

At the maturity of the mortgage, when the borrower decides on whether or not to make the final mortgage payment, the default option will be worthless if the house is worth more than the final payment and otherwise equal to the difference between the two. That is

$$D^{-}\left[H, w, t(I,6)\right] = \max\left[0, \left(\mathrm{PVMP}_{I,6} - H\right)\right] \tag{A.13}$$

On monthly payment dates other than the maturity, the default option value is adjusted for the difference between value of the remaining payments and the house price when there is default, and remains unchanged by the payment under conditions of no default.

$$\begin{aligned} &D^{-}\left[H, w, t(i,j)\right] = A^{-}\left[w, t(i,j)\right] - H \quad \text{if} \quad V^{-}\left[H, w, t(i,j)\right] = H \\ &\quad \text{(default)} \\ &D^{+}\left[H, w, t(i,j)\right] \quad \text{if} \quad V^{-}\left[H, w, t(i,j)\right] = V^{+}\left[H, w, t(i,j)\right] + \mathrm{PVMP}_{j} \\ &\quad \text{(no default)} \end{aligned} \tag{A.14}$$

At any adjustment date during the mortgage term, the value of default option is given by

$$\begin{aligned} &D^{-}\left[H, w, t(i-1,6)\right] = A^{-}\left[w, t(i-1,6)\right] - H \quad \text{if } V^{-}\left[H, w, t(i-1,6)\right] = H \\ &\quad \text{(default)} \\ &D^{+}\left[\frac{H}{\mathrm{OB}(i,0)}, w, t(i-1,6)\right] * \left[\mathrm{OB}(i,0)\right] \end{aligned}$$

$$\text{if}\quad V^{-}\left[H, w, t(i-1,6)\right] = V^{+}\left[\frac{H}{\mathrm{OB}(i,0)}, w, t(i-1,6)\right]\mathrm{OB}(i,0) + \mathrm{PVMP}_{i-1,6}$$

(no default)

(A.15)

Appendix 2: Convergence and Stability of Numerical Solution for the WIPM Valuation Model

Convergence of the explicit finite difference method depends on the size of the time-step, size of the space step, and size of the derivative coefficients. For a two state variable valuation problem, the general expression for the explicit difference equation can be written as follows

$$\frac{V_{i,j}^{n} - V_{i,j}^{n+1}}{\delta t} + a_{i,j}^{n}\left(\frac{V_{i+1,j}^{n} - 2V_{i,j}^{n} + V_{i-1,j}^{n}}{\delta H^{2}}\right) + b_{i,j}^{n}\left(\frac{V_{i+1,j}^{n} - V_{i-1,j}^{n}}{2\delta H}\right) + c_{i,j}^{n}V_{i,j}^{n} +$$

$$d_{i,j}^{n}\left(\frac{V_{i,j+1}^{n} - 2V_{i,j}^{n} + V_{i,j-1}^{n}}{\delta w^{2}}\right) + e_{i,j}^{n}\left(\frac{V_{i+1,j+1}^{n} - V_{i+1,j-1}^{n} - V_{i-1,j+1}^{n} + V_{i-1,j-1}^{n}}{4\delta H\delta w}\right) +$$

$$f_{i,j}^{n}\left(\frac{V_{i,j+1}^{n} - V_{i,j-1}^{n}}{2\delta w}\right) = O\left(\delta t, \delta H^{2}, \delta w^{2}\right) \qquad \text{(A.16)}$$

where $a, b, c, d, e,$ and f are all the coefficients in the main PDE. In order for a numerical solution of a PDE of the type under this study to converge and be stable, the following conditions must be satisfied (see Wilmott, 2000).

$$c_{i,j}^{n} \le 0 \qquad \text{(A.17)}$$

$$\frac{\left|b_{i,j}^{n}\right|}{a_{i,j}^{n}}\delta H + \frac{\left|f_{i,j}^{n}\right|}{d_{i,j}^{n}}\delta w \le 2 \qquad \text{(A.18)}$$

$$a_{i,j}^{n}\frac{\delta t}{\delta H^{2}} + d_{i,j}^{n}\frac{\delta t}{\delta w^{2}} \le \frac{1}{2} \qquad \text{(A.19)}$$

In financial problems, we almost have a negative c; often it is simply $-r$, where r is the risk-free interest rate. As Wilmott (2000) states, the second constraint on space step can be avoided by using an upwind differencing method. We have already used this method in order to improve stability of our numerical solution. The third constraint is a serious limitation on the size of the time-step (see Wilmott, 2000). Applying this constraint to our problem, Equation (A.17) becomes

$$\frac{1}{2}H(x)^2\,\sigma_H^2\omega^2x^4\left(\frac{k}{h^2}\right)+\frac{1}{2}w(y)\sigma_w^2\psi^2y^4\left(\frac{k}{l^2}\right)\le\frac{1}{2} \tag{A.20}$$

Since $h=l$, Equation (A.18) becomes

$$\frac{\left[\left(H(x)^2\,\sigma_H^2\omega^2x^4\right)+\left(w(y)\sigma_w^2\psi^2y^4\right)\right]k}{h^2}\le 1 \tag{A.21}$$

It is necessary to describe a time-step that is capable of providing a guarantee that the third condition will be valid for each point of the grid and each set of economic environment parameters. Following Kau et al. (1993) and Azevedo-Pereira et al. (2002), in order to achieve stability, we used 66 time steps a month that corresponds to $k=0.00\overline{126}$. Using the base parameter values, Table A1 presents the calculations for testing whether the time-step constraint holds. We chose the central point and the interior corners of the grid as a sample case.

Table A1: Measuring the stability of numerical solution with a sample of points (Scale factors ω is 0.75, ψ is 12.5. State variable volatilities σ_H is 0.10, σ_w is 0.15.)

Grid points				Numerical stability limitation
x	$H(x)$	y	$w(y)$	$\frac{\left[\left(H(x)^2\sigma_H^2\omega^2x^4\right)+\left(w(y)\sigma_w^2\psi^2y^4\right)\right]k}{h^2}$
0.98	0.03	0.02	3.92	1.9E-05
0.98	0.03	0.98	0.002	0.0167
0.02	65.3	0.02	3.92	1.9E-05
0.02	65.3	0.98	0.002	0.0167
0.50	1.333	0.50	0.08	0.0572