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## Abstract

It is well known that if income distribution  $y$  is majorized by income distribution  $x$  (*i.e.*  $y$  Lorenz-dominates  $x$  at constant sum, the welfare standard of  $y$  being more equally distributed than  $x$ ) then  $y$  can be reached from  $x$  by a succession of transfers “from the rich to the poor” that do not go beyond permuting their incomes. A number of results have been published that characterize special cases of such a system of transfers that are “simple” either in terms of number of transfers or of total amount transferred. We show that some of those results are flawed and we provide a number of new ones.

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# **REFINEMENTS OF MUIRHEAD'S LEMMA AND INCOME INEQUALITY**

**Ronny Aboudi**

Department of Management Science

University of Miami

P.O.B 248237

Coral Gables, FL 33124

U.S.A.

Tel: (305) 284 1966

Fax: (305) 284 2321

e-mail: [raboudi@miami.edu](mailto:raboudi@miami.edu)

**Dominique Thon**

Economics Programme

Department of Social Sciences

University of Macau,

Taipa,

Macao, China.

Tel: +853 3978462

Fax: +853 838312

e-mail: [thondomi@umac.mo](mailto:thondomi@umac.mo)

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## 1. Introduction

Let  $n$  be a number of persons and  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  be two allocations of income to those persons with  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ . An important concept in the economic literature on the distribution of a prescribed total amount of income between  $n$  persons (or income units) is the one of a T-transform, that is, a special  $n \times n$  bistochastic matrix of the form

(1.1)

$$T = \lambda Q + (1 - \lambda)I; \quad 0 \leq \lambda \leq 1,$$

where  $Q$  is a permutation matrix that interchanges two coordinates and  $I$  is the identity matrix.

The concept is due to Muirhead (1903); see Marshall and Olkin (1979, p. 21). The significance of a T-transform is that if  $x^* = xT$ , then  $x_k^* = (1 - \lambda)x_k + \lambda x_j$ ,  $x_j^* = (1 - \lambda)x_j + \lambda x_k$  and  $x_i^* = x_i$  for all  $i \neq j, k$ . If we assume that  $x_k \leq x_j$ , it is clear that  $x_k^*, x_j^* \in [x_k, x_j]$  and if  $\delta = \lambda(x_j - x_k)$ ,  $x_k^* = x_k + \delta$  and  $x_j^* = x_j - \delta$ , where  $0 \leq \delta \leq x_j - x_k$ . That is,  $\delta$  units of income are transferred from  $j$  to  $k$ , where the income for  $j$  is originally larger than the income for  $k$ , and one can think of a T-transform as performing between two persons a single transfer of income, which is such that the income of the recipient does not become strictly larger than the *original* income of the donor. We call such a transfer a Muirhead transfer and we use “T-transform” and “Muirhead transfer” interchangeably. If  $\lambda = 1$  in (1.1), then we have an extreme Muirhead transfer, corresponding to an extreme T-transform; clearly this is as large as such a transfer can be and the application of this extreme T-transform permutes two elements of the vector it multiplies. The

notation  $x_{(1)}, x_{(2)}, \dots, x_{(n)}$  represents the increasing rearrangement of  $x$ .

**Definition 1.1** The following well-known preorder over vectors in  $\mathfrak{R}^n$  (the opposite of majorization):

$$\sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)}; k = 1, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^n x_i = \sum_{i=1}^n y_i,$$

we represent by  $x \triangleleft y$ .

The fundamental relationship between the concept of a T-transform and the one of majorization is given in the following result (Marshall and Olkin (1979, B.1., p. 21)).

**Lemma 1.1 (Muirhead's Lemma)** If  $x \triangleleft y$ , then  $y$  can be reached from  $x$  by successive applications of a finite number of T-transforms.

The result was essentially proven by Muirhead (1903); see also Hardy, Littlewood and Polya (1952, p. 47). See *e.g.* Foster (1985), Sen (1997) for the Lemma's place in the theory of the measurement of income inequality. Marshall and Olkin (1979, pp. 21, 22)'s proof is the one we shall refer to. The proof proceeds by noting that, because the permutation of two elements in a vector of incomes is a Muirhead transfer (an extreme one), then one can assume that  $x$  and  $y$  are similarly ordered. Then without loss of generality, one can assume them to be increasingly ordered. The proof then proceeds by showing that, starting from  $x \uparrow$ , a finite number of Muirhead transfers, each one of which keeps the resulting vector increasingly ordered will lead us to  $y \uparrow$ , with  $x \uparrow$ ,  $y \uparrow$  representing the increasing rearrangements of  $x$  and  $y$ , respectively. From there,  $y$  itself can be reached from  $y \uparrow$  by the application of a finite sequence of extreme T-transforms, each of

which permutes two incomes. We illustrate this sequence on Figure 1, for  $n = 3$ . Let  $x = (4, 1, 13)$  be the initial income allocation and let  $y = (9, 6, 3)$ ;  $x \triangleleft y$ . Remembering that  $x \uparrow = (x_{(1)}, x_{(2)}, x_{(3)})$  and labeling the persons according to this increasing arrangement, one can reach  $x \uparrow$  by permuting the incomes of persons 1 and 2; then a transfer from person 3 to person 2 followed by a transfer from 2 to 1 bring us to  $y \uparrow$  along the path  $x \uparrow - A - y \uparrow$  in Figure 1. Those order-preserving transfers are the only kind explicitly considered in the proof (there are other possibilities, for example the paths  $x \uparrow - B - y \uparrow$  or  $x \uparrow - C - y \uparrow$ ). Once at  $y \uparrow$ , then  $y$  can be reached by a sequence of pair-wise permutations (*i.e.* extreme Muirhead transfers) for example the path  $y \uparrow - D - E - y$ . There is no reason to presume - and there are reasons to doubt - that the complete path  $x - x \uparrow - A - y \uparrow - D - E - y$  contains the minimum number of Muirhead transfers needed to reach  $y$  from  $x$ , but Muirhead's Lemma is merely concerned with existence of a finite sequence of Muirhead transfers if  $x \triangleleft y$ . One could think of some refinements of the result. By a refinement of Muirhead's Lemma, we mean a statement establishing not merely the existence of this sequence but furthermore proving that if  $x \triangleleft y$ , there exists some such sequence that satisfies special "nice" properties beyond merely being a finite sequence of Muirhead transfers. As far as we know, three such refinements have been proposed. The first one is Proposition 1.1 below, which is essentially Lemma 2.6 of Fields and Fei (1978). We say that  $x$  and  $y$  are similarly ordered if  $(x_i - x_j)(y_i - y_j) \geq 0, \forall i, j$ . If  $x$  and  $y$  are similarly ordered, then assuming that both are increasingly ordered involves no loss of generality, as it is simply a matter of labeling the elements of the vector, *i.e.* labeling the persons.

**Definition 1.2.** Given a pair of vectors  $x, y \in \mathfrak{R}^n$ , let  $d(x, y)$  denote the number of indices  $i$ , where  $x_i \neq y_i$ .

Obviously when  $x \neq y$ ,  $2 \leq d(x, y) \leq n$ , because  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ ,  $x$  can differ from  $y$  by two components but not by a single one.

**Proposition 1.1.** If  $x$  and  $y$  are similarly ordered and  $x \triangleleft y$ , then  $y$  can be reached from  $x$  by successive applications of at most  $[d(x, y) - 1]$  T-transforms.

Lemma 2.6 of Fields and Fei (1978) is actually somewhat weaker, but can easily be strengthened to Proposition 1.1. First, Fields and Fei (1978) assume unnecessarily that  $x, y$  are non-negative. Second, their upper bound on the number of T-transforms is unnecessarily large by one unit (it is  $d(x, y)$ ) simply because they overlook the fact that if  $x \neq y$ ,  $2 \leq d(x, y)$ . The following proposed refinement of Muirhead's Lemma (Marshall and Olkin (1979), B.1.a, p. 22; see also Theorem B.6, p. 24) does not assume  $x, y$  to be similarly ordered.

**Proposition 1.2.** If  $x \triangleleft y$ , then  $y$  can be reached from  $x$  by successive applications of at most  $n - 1$  T-transforms.

Now, careful consideration of the proof of this proposition in Marshall and Olkin (1979, B.1.a) shows that they actually consider only a sequence of Muirhead transfers from  $x \uparrow$  to  $y \uparrow$ , and prove Proposition 1.1 above, and *not* Proposition 1.2. In fact, Marshall and Olkin (1979) simply appeal to the proof of Muirhead's Lemma discussed above, that actually establishes Proposition 1.1, but not Proposition 1.2.

A third refinement of Muirhead's Lemma appears as a technical lemma in Bossert and Fleurbaey (2002, Lemma 2, p. 116).

**Proposition 1.3.** If  $x \triangleleft y$ , then  $y$  can be reached from  $x$  by successive applications of a finite number of T-transforms, and the sequence is such that, for  $i = 1, 2, \dots, n$ , if  $i$  is a donor in one of the transfers, he is the recipient in none, and vice versa.

Now, clearly, Proposition 1.2 does not follow from Proposition 1.1, and this leaves Proposition 1.2 an interesting conjecture, but, as indicated above, without a known proof. Is the condition that  $x$  and  $y$  be similarly ordered essential for the conclusion of Proposition 1.1 to hold? One can certainly find examples where it is not. Consider Figure 1 again and the path  $x \rightsquigarrow y$ . This path consists of  $2 = n - 1$  Muirhead transfers, and each one of them changes the ordering of the incomes. In this particular three-dimensional case, the conclusion of Proposition 1.2 holds. In Section 2 we prove that Proposition 1.2 is nevertheless generally false. This leaves Propositions 1.1 and 1.3 as the only established results - that we are aware of - that describe a simple path of Muirhead transfers from  $x$  to  $y$  if  $x \triangleleft y$  (with the considerable restriction that  $x, y$  are similarly ordered for Proposition 1.1).

Fields and Fei (1978), Marshall and Olkin (1979) and Bossert and Fleurbaey (2002) are all concerned with establishing the existence of a sequence of Muirhead transfers that is in some sense *simple*. There are two immediate points of view from which a sequence of transfers can be viewed as simple: the point of view of how many separate transfers there are in the sequence, and the one of how much income is transferred in total. Fields and Fei (1978) as well as Marshall and Olkin (1979) are concerned with the former; Bossert and Fleurbaey (2002) are indirectly

concerned with the latter, see Section 2 below. Thus Proposition 1.1, which is restricted to similarly ordered vectors, establishes the existence of a system of Muirhead transfers that is economical in terms of the *number of transfers*. Proposition 1.3 on the other hand describes a system of Muirhead transfers that is economical in terms of the total *amount transferred*. This economy follows from the fact that such a system of transfers contains only direct transfers from people richer in  $x$  than in  $y$  to people richer in  $y$  than in  $x$ .

In this paper, we consider both forms of simplicity and determine the conditions under which a system of transfers exhibits them. Section 2 demonstrates that in some very special cases where  $x \prec y$ , one can reach  $y$  from  $x$  in at most  $d(x, y) - 1$  Muirhead transfers, but we also show that Proposition 1.2 is generally false. Section 3 shows that the conclusion of Proposition 1.2, and more, follow if one assumes that  $x$  and  $y$  are related through a binary relation that is stronger than majorization. Section 4 provides a discussion of the problem from a broader perspective and in the context of flows-in-networks. Section 5 concludes. The purely technical results and the longer proofs are relegated to the Appendix.

## 2. Majorization.

We first state a very general definition of a system of transfers, and then a specialization thereof.

**Definition 2.1.** Let  $x, y \in \mathfrak{R}^n$  and  $\sum_1^n x_i = \sum_1^n y_i$ . An ***x-y system of transfers*** is a set of pair-wise transfers that allows one to reach  $y$  from  $x$ . The donor and recipient of a transfer are different people.

**Definition 2.2.** Let  $x, y \in \mathfrak{R}^n$  and  $\sum_1^n x_i = \sum_1^n y_i$ . A ***Muirhead x-y system of transfers*** is an  $x-y$  system of transfers (Def. 2.1) such that each transfer is a Muirhead transfer when the transfers are performed in some sequence.

Obviously, an  $x-y$  system of transfers produces  $y$  from  $x$  regardless of the order in which the transfers are performed. On the other hand to determine whether  $y$  can be reached from  $x$  through a sequence of Muirhead transfers requires the consideration of the order in which the transfers are performed. Note the general nature of the sets of transfers defined above; there is for example no presumption that there is a single transfer between a given pair of persons, nor that a person cannot be a donor in a transfer and a recipient in another, nor even, in Definition 2.1, that there cannot be multiple transfers both from  $j$  to  $k$  and from  $k$  to  $j$ .

The two properties of systems of transfers that we focus on are as follow. The first one was the concern of Fields and Fei (1978) and Marshall and Olkin (1979).

**Property NT (Number of transfers).** The number of transfers in the  $x-y$  system of transfers is strictly smaller than  $d(x, y)$ , the number of persons whose income

is different in  $x$  and  $y$ .

**Property D (Direct).** The  $x$ - $y$  system of transfers is such that if a person is the donor in one transfer, he is the recipient in none, and vice-versa.

Property D is the property that Bossert and Fleurbaey (2002) did consider in Proposition 1.3 above. Property D can be expressed in terms of minimizing the total amount transferred.

**Lemma 2.1.** Let  $x, y \in \mathfrak{R}^n$  and  $\sum_1^n x_i = \sum_1^n y_i$ . An  $x$ - $y$  system of transfers satisfies Property D if and only if the total amount transferred by the system is equal to  $\sum_{i=1}^n |x_i - y_i|/2$ . Furthermore, there is no  $x$ - $y$  system of transfers that transfers a smaller amount in total. (See Appendix.)

Note that Property D can also be interpreted as meaning that there is no “trans-shipment”, that is why it is called *direct*. Any income transferred goes directly to its final destination. If a person were to both receive and give, then some income would in effect be trans-shipped to someone else through him and thus, some income would be transferred twice. Property D precludes *inter alia* that income goes both from  $j$  to  $k$  and from  $k$  to  $j$ . It also precludes that a person whose income is the same in  $x$  and  $y$  would either receive or give. See Table A.2 in the Appendix for examples.

**Definition 2.3.** If  $x_i > y_i$  then person  $i$  is *richer in  $x$*  (than in  $y$ ). If  $x_i < y_i$  then person  $i$  is *richer in  $y$*  (than in  $x$ ).

It is convenient to partition the set of persons  $N = \{1, 2, \dots, n\}$  into  $R_x$  (the persons richer in  $x$ ),  $R_y$  (the persons richer in  $y$ ) and  $N_U$  (the persons with the same income in  $x$  and  $y$ ). Formally:  $R_x = \{i \mid x_i > y_i, 1 \leq i \leq n\}$ ,  $R_y = \{i \mid x_i < y_i, 1 \leq i \leq n\}$  and  $N_U = \{i \mid x_i = y_i, 1 \leq i \leq n\}$ . Note that those distinctions are entirely distinct from the notion of donors and recipients, as the latter are determined by the application of a specific set of transfers. Whether a person is in  $R_x$  for example depends only on the pair  $x, y$  and is independent of any system of transfers. A person in  $R_x$  can very well, in some system of transfers, be a donor in one transfer and a recipient in another one. Our first result is that the conclusion of Proposition 1.2 is true when there is exactly one person in  $R_x$  or exactly one in  $R_y$ .

**Theorem 2.1.** If  $x, y \in \mathfrak{R}^n$ ,  $x \triangleleft y$ , and either  $|R_x|=1$  or  $|R_y|=1$ , then there exists a Muirhead  $x$ - $y$  system of transfers that satisfies Property NT and Property D, in which the only donor is the person in  $R_x$  or the only recipient is the one in  $R_y$ , respectively. (See Appendix).

**Example 2.1.** Let  $n = 4$ ,  $x = (1, 14, 9, 7)$  and  $y = (3, 6, 10, 12)$ . Clearly  $|R_x|=1$ . We have that  $y$  can be reached from  $x$  by three Muirhead transfers: transfer 1 unit from person 2 to 3, 5 units from 2 to 4 and 2 units from 2 to 1. Note that the first recipient is the person in  $R_y$  who has the largest income in  $x$ , the second recipient is the person in  $R_y$  who has the second largest income in  $x$ , and so on.

**Example 2.2.** Let  $n = 4$ ,  $x = (5,10,2,14)$  and  $y = (3,6,10,12)$ . Clearly  $|R_y|=1$ . We have that  $y$  can be reached from  $x$  by three Muirhead transfers: transfer 2 units from 1 to 3, 4 units from 2 to 3 and 2 units from 4 to 2. Note that the first donor is the person in  $R_x$  who has the lowest income in  $x$ , the second donor is the person in  $R_x$  who has the second lowest income in  $x$ , and so on.

Theorem 2.1 has the following consequence, which shows that the path  $x - z - y$  constructed in Figure 1 is the rule rather than the exception.

**Corollary 2.1.** If  $x, y \in \mathfrak{R}^n$ , with  $d(x, y) \leq 3$  and  $x \triangleleft y$ , then there exists a Muirhead  $x$ - $y$  system of transfers that satisfies Property NT and Property D.

**Proof.** The result is obvious when  $d(x, y) = 0$ . Since the vectors  $x$  and  $y$  have the same sum,  $d(x, y) \neq 1$  and whenever  $d(x, y) \geq 2$ , we have that  $|R_x| \geq 1$  and  $|R_y| \geq 1$ . Since  $d(x, y) \leq 3$  and  $d(x, y) = |R_x| + |R_y|$ , then  $|R_x|=1$  or  $|R_y|=1$ . Hence the premises of Theorem 2.1 hold. ■

We now establish with a counterexample that Proposition 1.2 from Marshall and Olkin (1979) is in general false.

**Theorem 2.2.** If  $x^\diamond = (20, 15, 1, 5)$  and  $y^\diamond = (2, 4, 16, 19)$  then the number of T-transforms needed in order to reach  $y^\diamond$  from  $x^\diamond$  is at least  $4 = d(x, y)$ . (See Appendix.)

We have thus:

**Corollary 2.2.** Let  $x, y \in \mathfrak{R}^n$  and  $x \triangleleft y$ . It is not necessarily possible to reach  $y$  from  $x$  through the application of fewer than  $d(x, y)$  transfers.

**Example 2.3.** Note that  $y^\diamond$  can be reached from  $x^\diamond$  in four T- transforms. One such way is as follows, with  $j$  the donor,  $k$  the recipient and  $\delta$  the size of the transfer:

$x^\diamond = (20, 15, 1, 5)$ , let  $j = 2, k = 4, \delta = 10$ , to obtain

$x^1 = (20, 5, 1, 15)$ , let  $j = 2, k = 3, \delta = 1$ , to obtain

$x^2 = (20, 4, 2, 15)$ , let  $j = 1, k = 4, \delta = 4$ , to obtain

$x^3 = (16, 4, 2, 19)$ , let  $j = 1, k = 3, \delta = 14$ , to obtain

$x^4 = (2, 4, 16, 19)$ , which is  $y^\diamond$ .

This describes a Muirhead  $x$ - $y$  system of transfers which, according to Theorem 2.2, contains the smallest possible number of transfers for  $x^\diamond, y^\diamond$  as given.

We recapitulate the situation about Proposition 1.2. The proposition is in general false, but there are particular cases where its conclusion holds: if  $x$  and  $y$  are similarly ordered (Proposition 1.1) or if either  $|R_x|=1$  or  $|R_y|=1$  (which always holds if  $d(x, y) \leq 3$ ). Otherwise, taking note of the fact that it takes at most  $n-1$  extreme Muirhead transfers (*i.e.* simple permutations) to re-order any vector increasingly, and of Proposition 1.1, one obtains directly the following very conservative bound.

**Theorem 2.3.** Let  $x, y \in \mathfrak{R}^n$  and  $x \triangleleft y$ . Then  $y$  can be reached from  $x$  through a succession of at most  $2n - 2$  T-transforms.

Whether a better bound can be obtained in general for  $x \triangleleft y$  is an open question. In the next section we show that Proposition 1.1 can be generalized.

### 3. Second degree Pareto dominance

Consider the following binary relation, that we call second degree Pareto dominance.

**Definition 3.1.** Let  $x, y \in \mathfrak{R}^n$ . We say that  $x \triangleleft^{P^2} y$  if there exists a permutation  $\pi$  of  $N$  such that  $y_{\pi(1)} \leq y_{\pi(2)} \leq \dots \leq y_{\pi(n)}$ , and

$$\sum_{i=1}^k x_{\pi(i)} \leq \sum_{i=1}^k y_{\pi(i)}, \quad k = 1, \dots, n-1, \quad \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

We then say that  $y$  *Pareto dominates*  $x$  *by the second degree*.

If the values of the vector  $y$  are distinct, then  $\pi$  is unique. On the other hand, if  $y$  contains elements that are equal to each other, then there may exist several such permutations. Note that in order to check whether  $x \triangleleft^{P^2} y$  holds, one can sort the vector  $y$  increasingly and in the case of ties, use “ $x$  as a secondary key”, that is, for a set of indices where the  $y$  values are equal, choose a permutation where  $x$  is increasingly ordered. The inequalities of Definition 3.1 hold for the above rearrangement if and only if  $x \triangleleft^{P^2} y$ . For example, if  $y = (20, 10, 20)$  and  $x = (5, 7, 37)$ , then both  $(7, 5, 37)$  and  $(7, 37, 5)$  constitute rearrangements of  $x$  according to the increasing order of  $y$ , but only the first one is sorted by using  $x$  as a secondary key. In this case, only the first rearrangement satisfies both sets of inequalities of Definition 3.1.

This binary relation derives its name from the fact that it has the same relationship to constant-mean second degree stochastic dominance as the Pareto ordering has to first degree stochastic dominance (in what follows, “richer” is understood in the weak sense and constant sum is assumed wherever called for by a definition):

First degree dominance can be formulated as:

*The poorest person in  $y$  is richer than the poorest person in  $x$ .*

*The second poorest person in  $y$  is richer than the second poorest person in  $x$ .*

*etc.....*

The fact that  $y$  Pareto dominates  $x$  can be worded as:

*The poorest person in  $y$  is richer than this person is in  $x$ .*

*The second poorest person in  $y$  is richer than this person is in  $x$ .*

*etc.....*

Constant-mean second degree dominance/Lorenz dominance ( $x \triangleleft y$  in our case)

is:

*The poorest person in  $y$  is richer than the poorest person in  $x$ .*

*The two poorest persons in  $y$  are collectively richer than the two poorest persons in  $x$  are.*

*etc.....*

Our binary relation  $x \triangleleft^{P2} y$  on the other hand can be paraphrased:

*The poorest person in  $y$  is richer than this person is in  $x$ .*

*The two poorest persons in  $y$  are collectively richer than those two persons are in  $x$ .*

*etc.....*

Obviously  $\triangleleft^{P2}$  is, like the Pareto ordering, not symmetric, while  $\triangleleft$  is. The binary relation  $\triangleleft^{P2}$  is not transitive (unless  $x, y, z, \dots$  are all similarly ordered) as the following example shows:  $x = (0, 8, 14, 25); y = (2, 6, 14, 25); z = (12.5, 7, 13.5, 14)$ . It is easy to check that  $x \triangleleft^{P2} y$  and  $y \triangleleft^{P2} z$  hold while  $x \triangleleft^{P2} z$  does not. Thus while  $\triangleleft$  is a pre-order (a binary relation that is reflexive and transitive),  $\triangleleft^{P2}$  is merely a reflexive binary relation. It is clear that  $x \triangleleft^{P2} y$  implies  $x \triangleleft y$ , and that  $\{x \triangleleft y \text{ and } x,$

$y$  similarly ordered} implies  $x \triangleleft^{P^2} y$ .

The relationship  $x \triangleleft^{P^2} y$  is a natural way to express that, if one goes from  $y$  to  $x$ , and is concerned with equality but does not make the heroic assumption of anonymity (symmetry), income distribution  $x$  is worse than income distribution  $y$ .

We have the following result, that implies Proposition 1.1.

**Theorem 3.1.** Let  $x, y \in \mathfrak{R}^n$ . If  $x \triangleleft^{P^2} y$ , then there exists a Muirhead  $x$ - $y$  system of transfers that satisfies Property NT and Property D.

In order to establish Theorem 3.1 we will first prove Lemma 3.1 that deals with a single T-transform. Lemma 3.1 shows that after applying a particular T-transform to  $x$ , one can obtain a vector  $x^*$ , with  $x^* \triangleleft^{P^2} y$ , where the number of pairs of incomes with unequal values is reduced by at least one. One is then able to apply Lemma 3.1 repeatedly to prove the desired result. The proof is a modification of the argument appearing in the proof of Lemma B.1, Marshall and Olkin (1979, p. 21).

**Lemma 3.1.** If  $x, y \in \mathfrak{R}^n$  and  $x \triangleleft^{P^2} y$ ,  $x \neq y$ , then the vector  $x^*$  can be obtained from  $x$  by a single T-transform with the following properties:

(3.1) The transfer is from  $j$  to  $k$  with  $x_j > y_j$  and  $x_k < y_k$ .

(3.2)  $x_k^* = x_k + \delta$  and  $x_j^* = x_j - \delta$ , where  $0 < \delta \leq (x_j - x_k)/2$ ,  $x_j^* \geq y_j \geq y_k \geq x_k^*$ ,

and all other components remain the same (the transfer is thus a Muirhead transfer).

(3.3)  $d(x^*, y) \leq d(x, y) - 1$ .

(3.4)  $x^* \triangleleft^{P^2} y$ .

**Proof.** We assume *m.l.g.* that the indexing is done such that  $y$  is increasingly ordered and  $x$  and  $y$  satisfy the inequalities of Definition 3.1. Let  $j$  be the smallest index where  $x_j > y_j$ . The index  $j$  must exist, otherwise,  $x_i \leq y_i$  for all  $i$  and  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$  imply that  $x = y$ . Let  $k$  be the smallest index  $k \leq j-1$  where  $x_k < y_k$ . The index  $k$  must exist, otherwise, by choice of  $j$  we have  $x_i = y_i$  for  $i = 1, 2, \dots, j-1$  and  $x_j > y_j$  and these would imply that  $\sum_{i=1}^j x_i > \sum_{i=1}^j y_i$  which is not possible as  $x \triangleleft^{P^2} y$ . By choice of  $j$  and  $k$  it is clear that (3.1) holds. Note that by choice of  $k$  and  $j$  and the fact that  $y$  is increasing, we have  $x_k < y_k \leq y_j < x_j$ . Let  $\delta = \min(x_j - y_j, y_k - x_k)$ . By choice of  $j$  and  $k$  it is clear that  $\delta > 0$ . Consider the transfer of  $\delta$  units from  $j$  to  $k$ , specifically, let:

$$x_i^* = x_i \text{ for } i \neq j, k; \quad x_k^* = x_k + \delta \text{ and } x_j^* = x_j - \delta.$$

By choice of  $\delta$  we have that  $x_k^* = x_k + \delta \leq x_k + y_k - x_k = y_k$

and  $x_j^* = x_j - \delta \geq x_j - (x_j - y_j) = y_j$ . But since  $k \leq j-1$  and  $y$  is increasingly

ordered, we have  $y_j \geq y_k$ . Thus, combining the last three inequalities we obtain  $x_j^* \geq y_j \geq y_k \geq x_k^*$ . Note that  $x_j^* \geq x_k^*$  is equivalent to  $\delta \leq (x_j - x_k)/2$ . Therefore, all the conditions of (3.2) hold. Note that if  $\delta = y_k - x_k$ ,  $x_k^* = y_k$  and if  $\delta = x_j - y_j$ ,  $x_j^* = y_j$ . As  $x_j > y_j$  and  $x_k < y_k$ , we have  $d(x^*, y) \leq d(x, y) - 1$ , which is (3.3). Finally we have to show that  $x^* \triangleleft^{p^2} y$ , (3.4). This can be done directly. By choice of  $j$  and  $k$  one has that:

$$x_i = y_i \text{ for } i = 1, 2, \dots, k-1,$$

$$x_i \leq y_i \text{ for } i = k, k+1, \dots, j-1.$$

Therefore,

$$\sum_{i=1}^r x_i^* = \sum_{i=1}^r x_i \leq \sum_{i=1}^r y_i \text{ for } r = 1, 2, \dots, k-1.$$

As  $x_i = y_i$  for  $i = 1, 2, \dots, k-1$  and  $x_k + \delta \leq y_k$ ,

$$\sum_{i=1}^k x_i^* = \left( \sum_{i=1}^{k-1} x_i \right) + x_k + \delta \leq \sum_{i=1}^k y_i.$$

By the previous inequality and the fact that  $x_i \leq y_i$  for  $i = k+1, \dots, j-1$ ,

$$\sum_{i=1}^r x_i^* = \sum_{i=1}^k x_i^* + \sum_{i=k+1}^r x_i \leq \sum_{i=1}^k y_i + \sum_{i=k+1}^r y_i \text{ for } k+1 \leq r \leq j-1.$$

Since  $\delta$  units are transferred from  $j$  to  $k$ ,  $k < j$ , we have that  $\sum_{i=1}^r x_i^* = \sum_{i=1}^r x_i$  for  $r \geq j$ , and thus:

$$\sum_{i=1}^r x_i^* = \sum_{i=1}^r x_i \leq \sum_{i=1}^r y_i \text{ for } r = j, j+1, \dots, n-1 \text{ and}$$

$$\sum_{i=1}^n x_i^* = \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

It is clear that the current indexing of  $x^*$  and  $y$  satisfies the inequalities of Definition 3.1.

**Example 3.1.** Let  $x = (2,9,3,23,1,71)$  and  $y = (4,8,15,20,30,32)$ . It is clear that  $d(x,y) = 6$ . According to the procedure of Lemma 3.1,  $j = 2$ ,  $k = 1$ ,  $\delta = \min(9-8, 4-2) = 1$  and thus  $x^* = (3,8,3,23,1,71)$  and  $d(x^*,y) = 5$ . One can readily check that  $x^* \triangleleft^{P^2} y$ .

Theorem 3.1 now easily follows from Lemma 3.1.

**Proof of Theorem 3.1.** Lemma 3.1 guarantees that whenever  $x \triangleleft^{P^2} y$  and  $x \neq y$ , there exists a Muirhead transfer that results in a vector  $x^*$  with the property that  $d(x^*,y) \leq d(x,y) - 1$  and  $x^* \triangleleft^{P^2} y$  (properties (3.2), (3.3) and (3.4)). Since  $x^* \triangleleft^{P^2} y$ , unless  $x^* = y$  the procedure in Lemma 3.1 can be applied starting with  $x^*$  to obtain another Muirhead transfer and a new vector. Since the number of differences is reduced by at least one each time the procedure is applied, and since  $x' = y$  if and only if  $d(x',y) = 0$ , then the total number of applications is bounded by  $d(x,y) - 1$ . This is due to the fact that  $x'$  and  $y$  can never differ by exactly one component. Property (3.2) of Lemma 3.1 guarantees that in all the transfers performed on intermediate vectors  $x'$ , the donor  $j$  always has more income than he does in  $y$ , until his income reaches the value  $y_j$ . Similarly, in such a transfer, the recipient  $k$  always has less income than in  $y$  until his income becomes  $y_k$ . Combining those facts with Property (3.1) we have that a person is never donor in one transfer and recipient in another one; thus Property D is satisfied.

**Example 3.2.** Consider  $x$  and  $y$  in Example 3.1. We apply the procedure in Lemma 3.1 iteratively to reach  $y$  from  $x$  through with the following transfers, with  $j$  the donor,  $k$  the recipient and  $\delta$  the size of the transfer:

$$\text{Step 1: } j=2, k=1, \delta=1, \quad x^1 = (3,8,3,23,1,71), \quad d(x^1, y) = 5.$$

$$\text{Step 2: } j=4, k=1, \delta=1, \quad x^2=(4,8,3,22,1,71), \quad d(x^2, y)=4.$$

$$\text{Step 3: } j=4, k=3, \delta=2, \quad x^3=(4,8,5,20,1,71), \quad d(x^3, y)=3.$$

$$\text{Step 4: } j=6, k=3, \delta=10, \quad x^4=(4,8,15,20,1,61), \quad d(x^4, y)=2.$$

$$\text{Step 5: } j=6, k=5, \delta=29, \quad x^5=(4,8,15,20,30,32), \quad d(x^5, y)=0.$$

The condition  $x \triangleleft^{P^2} y$  is a special case of majorization but it is independent of the conditions appearing in Section 2. The sufficient conditions of Section 2 (that majorization holds with either  $|R_x|=1$  or  $|R_y|=1$ ), do not imply  $x \triangleleft^{P^2} y$  as the following simple example shows:  $x = (10, 20); y = (16, 14)$ .

To summarize: the following three conditions are each sufficient for the existence of a Muirhead  $x \succ y$  system of transfers that satisfies both Property NT and Property D:  $x \triangleleft y$  and  $|R_x|=1$ ;  $x \triangleleft y$  and  $|R_y|=1$ ;  $x \triangleleft^{P^2} y$ . Proposition 1.1 is superseded by Theorem 3.1.

#### 4. Discussion.

The transformation of some  $n$ -vector  $x$  into  $n$ -vector  $y$  at constant sum can easily be represented in a flows-in-network framework. In this section we look at the results of the previous sections from that point of view. We broaden the perspective and start with the more general set-up of considering transfers ( *i.e.* flows) that need not satisfy the condition that they are Muirhead transfers (see Definition 2.1) and we are first interested in the minimal number of transfers needed to reach  $y$  from  $x$ . We show that our results of the previous sections on this matter are closely related to this problem.

From the theory of network flows, we shall specifically mobilize the Minimum Cost Network Flow Problem (MCNFP) and relate it to systems of transfers. We shall in particular show that the upper bound of  $d(x, y) - 1$  transfers that appears in several of our results is the best possible one in general. The MCNFP formulation of transfer systems also provides an alternate way of proving Theorem 2.2. In addition, Mixed Integer Programming (MIP) formulations for addressing explicitly the problem of finding the minimum number of transfers for specific instances of  $x$  and  $y$  are also presented. The results from the theory of networks will be quoted without proof. Refer to Hillier and Lieberman (2001) for a detailed discussion of the MCNFP, the Transportation Problem, MIP and the Fixed Charge Problem.

Definition 2.1 of  $x$ - $y$  system of transfers is very broad, and in fact, it allows for an infinite number of transfers by not excluding the possibility of performing many transfers between a particular pair of persons. To simplify the discussion we introduce the notion of a *simple*  $x$ - $y$  system of transfers.

**Definition 4.1.** An  $x$ - $y$  system of transfers is *simple* if for any  $i \neq j$  it contains at most one transfer from  $i$  to  $j$ .

Whenever there are multiple transfers from  $i$  to  $j$  one can sum the amounts of the individual transfers and replace the set of transfers by a single transfer and obtain a simple  $x$ - $y$  system of transfers. The transformed system will transfer the same amount of income but will perform the same or a smaller number of transfers.

The (uncapacitated) MCNFP considers the shipment of a single commodity in a network with  $n$  nodes and a set of arcs  $A$ . An arc from node  $i$  to node  $j$  is represented by the ordered pair  $(i, j)$ . In general  $A$  is a subset of the set of all possible ordered pairs  $(i, j)$  where  $i, j \in N$ . Each node has supply  $s_i$ ,  $i = 1, 2, \dots, n$  (demand is represented by a negative number) where  $\sum_1^n s_i = 0$ . One is able to ship a non-negative amount  $z_{ij}$  at a unit cost  $c_{ij}$  on arcs  $(i, j) \in A$ . The  $z_{ij}$  are known as decision variables and their values are known as flows. The objective of the problem is to find the flows that minimizes the total cost  $\sum_{(i,j) \in A} c_{ij} z_{ij}$  subject to the conditions that the flow at each node is balanced, that is, total flow out of node  $i$  minus the total flow into node  $i$  must equal the supply at node  $i$  or,

$$(4.1) \quad \sum_{k:(i,k) \in A} z_{ik} - \sum_{k:(k,i) \in A} z_{ki} = s_i \quad \text{for } i = 1, 2, \dots, n.$$

Finding a simple  $x$ - $y$  system transfers can be formulated as a MCNFP on  $n$  nodes. We call our formulation  $TS(x, y)$ . For  $i \in N$ , node  $i$  has supply  $s_i = x_i - y_i$ , it is clear

that the total supply is zero. The set of arcs is defined to be  $A = \{(i, j) \mid i, j \in N, i \neq j\}$ . We are interested in establishing the correspondence between  $x$ - $y$  systems transfers and feasible solutions to  $TS(x, y)$  so the objective function plays no role here and therefore we can set the unit cost to be zero for all arcs in  $A$ . We can establish the equivalence between feasible solutions to  $TS(x, y)$  and simple  $x$ - $y$  system of transfers. First we note that  $z_{ij}$ , the flow on arc  $(i, j) \in A$  is nonnegative and involves two different nodes, so it corresponds to a transfer from person  $i$  to person  $j$ . Since for  $i \in N$ ,  $s_i = x_i - y_i$ , equation (4.1) can be restated as:

$$(4.2) \quad y_i = x_i - \sum_{k \in N \setminus \{i\}} z_{ik} + \sum_{k \in N \setminus \{i\}} z_{ki}$$

that is, starting with income  $x_i$  for person  $i$  and after accounting for all the transfers from  $i$  and transfers to  $i$ , person  $i$  is left with income  $y_i$ . Thus, the flows on the arcs  $(i, j) \in A$  of any feasible solution of  $TS(x, y)$  correspond to a simple  $x$ - $y$  system of transfers. Conversely, given a simple  $x$ - $y$  system of transfers we can set the flow on arc  $(i, j) \in A$  to the amount transferred from  $i$  to  $j$ . Note that  $s_i > 0$  if and only if  $x_i > y_i$ . Since a simple  $x$ - $y$  system can always be found (one can always transfer all of  $x_i - y_i$  from all  $i$ 's with  $x_i > y_i$  to person 1 and then transfer the required amounts to persons 2 to  $n$ ),  $TS(x, y)$  is always feasible.  $TS(x, y)$  for  $x$  and  $y$  of Theorem 2.2 is shown in **Figure 2**.

We now illustrate several solutions to  $TS(x, y)$  for a number of  $x$ - $y$  pairs. The solutions will be represented by an  $n \times n$  matrix  $Z = (z_{ij})$ . Variables with value zero are generally omitted. The ones that are included serve only to distinguish among the rows and columns of  $Z$ . We will refer to these solutions in the discussion that follows. A summary of their properties is given in Table 4.1.

**Example 4.1.** Let  $n = 4$ ,  $x = (20,15,1,5)$  and  $y = (2,4,16,19)$ , as in Theorem 2.2. The system of transfers given in Example 2.3 corresponds to the following solution of  $TS(x,y)$ , depicted in **Figure 3**.

$$Z_1 = \begin{pmatrix} 0 & 0 & 14 & 4 \\ & & 1 & 10 \end{pmatrix}$$

In addition, we give two other feasible solutions of  $TS(x,y)$ :

$$Z_2 = \begin{pmatrix} 0 & 18 & & \\ & & 15 & 14 \end{pmatrix} \text{ and } Z_3 = \begin{pmatrix} 0 & 0 & 15 & 3 \\ & & & 11 \end{pmatrix}.$$

$Z_2$  corresponds to the system of transfers of 18 units from 1 to 2, 15 units from 2 to 3 and 14 units from 2 to 4;  $Z_3$  corresponds to the system of transfers of 15 units from 1 to 3, 3 units from 1 to 4 and 11 units from 2 to 4.

**Example 4.2.** Let  $n = 4$ ,  $x = (1,5,10,12)$  and  $y = (4,6,8,10)$ . We consider 3 different solutions of  $TS(x,y)$ :

$$Z_4 = \begin{pmatrix} & & 0 \\ & 0 & \\ 2 & & \\ 1 & 1 & \end{pmatrix}, \quad Z_5 = \begin{pmatrix} & & 0 \\ & 0 & \\ 3 & 1 & \\ & & 2 \end{pmatrix} \text{ and } Z_6 = \begin{pmatrix} & & 0 \\ & & \\ 1.5 & 0.5 & \\ 1.5 & 0.5 & \end{pmatrix}.$$

**Example 4.3.** Let  $n = 4$ ,  $x = (4,1,7,17)$  and  $y = (3,6,8,12)$ . We consider one solution of  $TS(x,y)$ :

$$Z_7 = \begin{pmatrix} & 1 & & \\ & & 0 & \\ & & & 0 \\ 0 & 4 & 1 & \end{pmatrix}$$

(transfer 1 unit from 1 to 2, 4 units from 4 to 2 and 1 units from 4 to 3). Performing the transfers in this order yields a sequence of three Muirhead

transfers.

We first focus on Property NT and start with general  $x$ - $y$  systems of transfers (Definition 2.1). Our first result is elementary.

**Proposition 4.1.** Let  $x, y \in \mathfrak{R}^n$  and  $\sum_1^n x_i = \sum_1^n y_i$ . There exists an  $x$ - $y$  system of transfers that satisfies Property NT.

**Proof.** W.l.g. we can assume that all persons with unchanged income are deleted a priori and thus  $n = d(x, y)$ . It is well known that if (uncapacitated) MCNFP has  $n$  nodes then a feasible solution is *basic* if and only if it contains at most  $n-1$  arcs with positive flow and this set of arcs does not contain a cycle. Furthermore, one can always find an optimal solution that is basic. Since  $TS(x, y)$  is feasible and bounded, it has an optimal solution so the result holds. ■

Solutions  $Z_2, Z_3, Z_4, Z_5$  and  $Z_7$  satisfy Property NT and do not contain a cycle. Proposition 4.1 shows the existence of an  $x$ - $y$  system of transfers with at most  $d(x, y)-1$  transfers. In fact, in most situations, the number of transfers is exactly  $d(x, y)-1$  and it falls below this bound only when *degeneracy* is present. In this formulation, degeneracy is equivalent to having a proper subset  $L \subseteq N$  where the total supply in  $L$  sums to zero (sum of initial and final income for persons in  $L$  are equal). If such a subset exists, the problem can be partitioned into two, considering the two sets  $L$  and  $N \setminus L$  separately and thus reducing the number of transfers by one. An example of degeneracy is as follows:  $x = (10, 20, 30, 40); y = (11, 19, 31, 39)$ . Obviously  $y$  can be reached from  $x$  in only two steps, by transferring, for example, one unit from 2 to 1 and the same from 4 to 3, but we

have in fact two problems:  $x^A = (10,20)$ ;  $y^A = (11, 19)$  and  $x^B = (30,40)$ ;  $y^B = (31, 39)$ . However, when degeneracy is not present, the minimum number of transfers is exactly  $d(x,y)-1$ , so that the result is the best possible general one.

Since the transfers in a Muirhead system are more restrictive than the ones we have considering, we can conclude that the above bound is the best possible one for Muirhead transfers as well. It is not surprising that the propositions in Sections 1, 2 and 3 all consider this bound. However, as shown in Corollary 2.2, in general the condition  $x \triangleleft y$  is not sufficient to guarantee the existence of a Muirhead  $x$ - $y$  system of transfer that satisfies Property NT. Theorem 2.1 shows that property NT is satisfied by some Muirhead  $x$ - $y$  system of transfer if  $x \triangleleft y$  and either  $|R_x|=1$  or  $|R_y|=1$ ; Theorem 3.1 shows that it is also satisfied if  $x \triangleleft^{P^2} y$  (see Solutions  $Z_4$  and  $Z_5$ ), but certainly neither of those conditions is necessary, as shown by Solution  $Z_7$  of Example 4.3, where  $x \triangleleft y$  holds but none of those conditions does.

It is interesting to note that the exclusion of cycles from system of transfers implies that the total number of transfers is at most  $d(x,y)-1$ . This follows from properties of trees and forests (graphs with no cycles).

**Corollary 4.1.** Let  $B \subseteq \{1,2,\dots,n\}$  with  $q$  elements. An  $x$ - $y$  system of transfers produced by Theorem 2.1 or by Theorem 3.1 contains at most  $q-1$  transfers  $(i, j)$  where  $i$  and  $j$  are in  $B$  (known as B-transfers).

**Proof.** Let  $B \subseteq \{1,2,\dots,n\}$  with  $q$  elements. Note that whenever a transfer is

performed, one of the persons involved will reach his income in  $y$  so that he will no longer be involved in future transfers. Suppose the system contains  $q-1$  B-transfers. After we perform  $q-1$  B-transfer,  $B$  will contain at most one person who is eligible to participate in future transfers, thus there will be no more B-transfers. ■

To illustrate, consider the set of transfers in Example 3.2. If  $B = \{3,4,5,6\}$ , then the system of transfers in this example contains three B-transfers. Note that the conclusion of Corollary 4.1 is equivalent to excluding cycles, so in effect the procedures given in Theorems 2.1 and 3.1 produce sets of cycle-free Muirhead transfers that satisfy property NT.

Recall that the pair  $x-y$  in Example 4.1 was the one used to provide a counter-example to Proposition 1.2. Solutions  $Z_2$  and  $Z_3$  are examples of  $x-y$  systems of transfers that contain  $3 = n-1$  transfers. However, they are not Muirhead  $x-y$  systems of transfers. An alternate approach to proving Theorem 2.2 is to enumerate all basic feasible solutions of  $TS(x,y)$  and show (by enumerating all  $3!$  permutations) that none of them can generate a system of Muirhead transfers. Table A.2 in the Appendix lists all 16 basic feasible solutions of  $TS(x,y)$ , none of which can give a system of Muirhead transfers.

We now consider Property D as well. Solutions  $Z_1, Z_3, Z_4, Z_6$  and  $Z_7$  satisfy this property. It is clear that in any  $x-y$  system of transfers a person who is richer in  $x$  must be a donor in at least one of the transfers and similarly a person who is richer in  $y$  must be a recipient at least once. Therefore, in order for Property D to hold one must impose conditions a priori on the set of transfers so that a person who is

richer in  $x$  can only be a donor and a person who is richer in  $y$  can be only a recipient. This can be accomplished by formulating a MCNFP called  $TP(x,y)$ . Let the set of nodes be  $N$ , with supply  $s_i = (x_i - y_i)$ ,  $i = 1, 2, \dots, n$ . It is clear that  $s_i > 0$  for  $i \in R_x$ ;  $s_i < 0$  for  $i \in R_y$  and the total supply sums to zero. The only arcs that are included are from persons in  $R_x$  to persons in  $R_y$ , so here the set of arcs is  $A = \{(i, j) \mid i \in R_x, j \in R_y\}$ . The unit cost again is irrelevant and can be set to zero.

**Proposition 4.2.** Let  $x, y \in \mathfrak{R}^n$  and  $\sum_1^n x_i = \sum_1^n y_i$ . Then there exists a  $x$ - $y$  system of transfers that satisfies Property NT and Property D.

**Proof.** We can assume *m.l.g.* that all persons with equal income have been deleted and that  $n = d(x, y)$ . Note that  $TP(x, y)$  is the MCNFP formulation for the standard (uncapacitated) Balanced Transportation Problem with suppliers in the set  $R_x$  and customers in the set  $R_y$ . It is well known that this transportation problem is always feasible and thus  $TP(x, y)$  is feasible and by construction of  $A$ , every feasible solution satisfies Property D. Furthermore, by employing the same arguments as in Proposition 4.1,  $TP(x, y)$  has a feasible solution that satisfies Property NT and the set of arcs of the solution does not contain a cycle.

Solutions  $Z_3, Z_4$  and  $Z_7$  satisfy both properties NT and D. Note that requiring the  $x$ - $y$  system of transfers to satisfy Property D in addition to Property NT does not affect the bound on the total number of transfers. The bound  $d(x, y) - 1$  is still the best possible general one.

Looking back, one can see that our results of Section 2 basically show that adding to the premises of Proposition 4.2 the condition that  $x \triangleleft y$  does not allow one to strengthen its conclusion by replacing “ $x$ - $y$  system of transfers” by “Muirhead  $x$ - $y$  system of transfers”. However Theorems 2.1 and 3.1 show that some stronger conditions are sufficient.

Properties NT and D are in general independent properties of  $x$ - $y$  systems of transfers. For example solutions  $Z_1$  and  $Z_6$  satisfy D but not NT, solutions  $Z_2$  and  $Z_5$  satisfy NT but not D, and finally, solution  $Z_3, Z_4$  and  $Z_7$  satisfy both properties. For the sake of completeness, we now establish the stronger fact that they are independent properties of Muirhead  $x$ - $y$  systems of transfers even when  $x \triangleleft^{P^2} y$  holds. Solution  $Z_6$  is an example where Property D holds but not Property NT, solution  $Z_5$  is an example where it is the opposite. In both cases,  $x \triangleleft^{P^2} y$  holds and the solution is a Muirhead  $x$ - $y$  system of transfers. Note that solution  $Z_7$  (Example 4.3) describes a Muirhead  $x$ - $y$  system of transfers that satisfies Properties NT and D but the conditions of neither Theorem 2.1 nor Theorem 3.1 are satisfied. This illustrates that while sufficient, they are not necessary.

Table 4.1

Solution	“ $x \triangleleft^{P2} y$ ” or “ $x \triangleleft y$ and either $ R_x =1$ or $ R_y =1$ ”	Property D	Property NT	Muirhead
$Z_1$	No	Yes	No	Yes
$Z_2$	No	No	Yes	No
$Z_3$	No	Yes	Yes	No
$Z_4$	Yes $[x \triangleleft^{P2} y]$	Yes	Yes	Yes
$Z_5$	Yes $[x \triangleleft^{P2} y]$	No	Yes	Yes
$Z_6$	Yes $[x \triangleleft^{P2} y]$	Yes	No	Yes
$Z_7$	No	Yes	Yes	Yes

We conclude this section by considering the direct strategy of determining exactly the minimum number of  $x$ - $y$  transfers needed to reach  $y$  from  $x$ . This approach is meant to give the exact number for a particular pair  $x, y$ ; for example, it would for example give a number of transfers smaller than  $d(x, y) - 1$  if there is any degree of degeneracy. We then consider Muirhead  $x$ - $y$  systems of transfers, assuming  $x \triangleleft y$ . Note that for Muirhead transfers degeneracy (as defined above) may not necessarily play a role. In both cases, we use a MIP (Mixed Integer Programming) formulation.

For a given pair  $x, y$ , finding an  $x$ - $y$  system of transfers can be accomplished by solving  $TS(x, y)$ . This can be done efficiently using the network simplex algorithm and the optimal solution will satisfy Property NT. However, when degeneracy is present, there is no guarantee that the solution will have the minimum number of

transfers. Note that detecting degeneracy in advance and partitioning  $N$  into the maximum number of subsets where each subset has total supply zero is computationally complex (in computational complexity terms it is known as NP-hard; see Garey and Johnson (1979)). The (NP-hard) problem of finding an  $x$ - $y$  system of transfer with the smallest feasible number of transfers can be formulated as a MIP known as the Fixed Charge Network Flow Problem. We consider the network defined for  $TS(x,y)$  [or  $TP(x,y)$  when requiring Property D]. For each  $(i,j) \in A$  a binary variable  $w_{ij}$  is introduced. It will have the value 1 if there a positive flow from  $i$  to  $j$  and will have the value 0 otherwise. The formulation  $FCTS(x,y)$  is:

$$\text{Min } \sum_{(i,j) \in A} w_{ij}$$

subject to:

$$\sum_{k:(i,k) \in A} z_{ik} - \sum_{k:(k,i) \in A} z_{ki} = s_i \text{ for } i = 1, 2, \dots, n$$

$$z_{ij} \leq Mw_{ij} \text{ for } (i,j) \in A$$

$$z_{ij} \geq 0 \text{ for all } (i,j) \in A \text{ and } w_{ij} \in \{0,1\} \text{ for } (i,j) \in A,$$

where  $M$  is a large number (for example  $M = \sum_{i=1}^n |x_i - y_i|$ ). The objective function counts the number of arcs with positive flow. The first constraint guarantees that the flow is balanced (see MCNFP) and the remaining ones insure that the flow on arc  $(i,j) \in A$  can be positive only if  $w_{ij}=1$ , (which is then counted in the objective function). Note that  $FCTS(x,y)$  is a MIP and solving it to optimality is much more time consuming than doing so for  $TS(x,y)$ .

We now turn to the case of majorization. Although solving  $FCTS(x,y)$  provides a lower bound for the number of Muirhead transfers, we present a MIP formulation



Consider the MIP formulation  $MNT(x, y, A)$ :

$$\text{Min } \sum_{k \in K} \sum_{(i, j) \in A} w_{ijk}$$

Subject to:

$$(4.3) \quad \sum_{(i, j) \in A} w_{ijk} \leq 1 \quad \text{for } k \in K$$

$$(4.4) \quad t_{ijk} \leq M w_{ijk} \quad \text{for } (i, j) \in A \text{ and } k \in K$$

$$(4.5) \quad t_{ijk} \leq M(1 - w_{ijk}) + z_{i, k-1} - z_{j, k-1} \quad \text{for } (i, j) \in A \text{ and } k \in K$$

$$(4.6) \quad z_{jk} = z_{j, k-1} + \sum_{r: (r, j) \in A} t_{rjk} \quad \text{for } j \in N \text{ and } k \in K$$

$$(4.7) \quad z_{ik} = z_{i, k-1} - \sum_{r: (i, r) \in A} t_{irk} \quad \text{for } i \in N \text{ and } k \in K$$

$$w_{ijk} \in \{0, 1\} \text{ and } t_{ijk} \geq 0 \quad \text{for } (i, j) \in A \text{ and } k \in K$$

$$z_{ik} \text{ unrestricted} \quad \text{for } i \in N \text{ and } k \in K \setminus \{2n - 2\}.$$

Constraints (4.3) guarantee that at each step at most one transfer is performed. Constraints (4.4) insure that if the pair  $(i, j)$  is not chosen as the transfer pair in step  $k$ , no amount will be transferred from  $i$  to  $j$ . If at step  $k$ , the transfer from  $i$  to  $j$  is chosen, then  $w_{ijk} = 1$ . Substituting into (4.5) we obtain the upper bound for this transfer, namely, the difference between income  $i$  and income  $j$  of the vector in step  $k-1$ . Since  $t_{ijk} \geq 0$ , (4.5) forces  $w_{ijk} = 0$  whenever the income of person  $i$  is smaller than that of person  $j$  (in step  $k-1$ ). Equations (4.6) and (4.7) guarantee that the sequence of Muirhead transfers starts with  $x$ , ( $k=0$ ) and terminates with  $y$  ( $k=2n-2$ ). Also, at each intermediate step, (4.6) and (4.7) compute the vector of incomes after the  $k^{\text{th}}$  transfer. The objective function

minimizes the total number of transfers. Note that since the objective function minimizes the sum  $w_{ijk}$ , transfers of value zero will not occur in any optimal solution. Whenever  $x \triangleleft y$ ,  $MNT(x, y, \mathcal{A})$  is feasible. However, the number of binary variables is  $O(n^3)$ , rendering it challenging to solve.

## 5. Conclusion

Muirhead's Lemma is central to the foundations of the measurement of income inequality. It says that if income distribution  $y$  Lorenz-dominates income distribution  $x$  and both distributions contain the same amount of total income ( $x$  majorizes  $y$ ), then  $y$  can be reached from  $x$  through a finite number of transfers from richer to poorer persons that do not go beyond a permutation. We call such transfers Muirhead transfers. The Lemma is fundamental to validate basic ideas in the evaluation of income distributions, for example the suitability of comparing Lorenz curves to rank income distributions or the use of Schur convex functions to measure income inequality.

The economic literature has contributed two refinements to Muirhead's Lemma. Fields and Fei (1978) essentially showed that, if  $x$  and  $y$  are similarly ordered, then there exists a sequence of Muirhead transfers leading from  $x$  to  $y$  that contains at most one fewer transfer than there are persons whose income is different in  $x$  and in  $y$  (Property NT). Bossert and Fleurbaey (2002) showed that if  $x$  majorizes  $y$ , there exists a sequence of Muirhead transfers leading from  $x$  to  $y$  that contains only transfers from people richer in  $x$  than in  $y$  to people richer in  $y$  than in  $x$ , or equivalently, a sequence in which no-one is both a donor and a recipient (Property D). Such a sequence of transfers can also be shown to minimize the total amount of income transferred.

Marshall and Olkin (1979) claimed that the conclusion of Fields and Fei (1978) follows even without the assumption that  $x$  and  $y$  are similarly ordered. An examination of Marshall and Olkin (1979)'s proof shows that what they actually establish is in effect equivalent to Fields and Fei (1978)'s result, *i.e.* they assume in

their proof that  $x$  and  $y$  are similarly ordered. We provide a counter-example for  $n = 4$  to show that it is not simply a matter of the claim being in need of a proof: the claim is false.

We present three sets of new sufficient conditions for the existence of a sequence of Muirhead transfers that satisfies Property NT. In particular, we show that a sufficient condition for Property NT to hold is that there exists between  $x$  and  $y$  a binary relation that we call second-degree Pareto dominance. It seems to have some interest of its own and to be worth studying further. The result of Fields and Fei (1978) becomes a particular case of ours. Note that we further show that if second degree Pareto dominance holds, then there exists a sequence of Muirhead transfers that satisfies *both* Property NT and Property D. The earlier results dealt with one of those properties only.

Finally, we re-visit the problem of characterizing systems of transfers in a broader perspective, *i.e.* without as well as with majorization, with the tools of flows-in-networks theory, and we show that the bound on the number of transfer that we have reached is the best possible general one. Those tools further give us an alternative way of proving our counterexample to Marshall and Olkin (1978)'s claim; they are also used to show that our conditions are sufficient but not necessary.

## APPENDIX

### Proof of Lemma 2.1.

We assume that all persons with unchanged income are deleted a priori. Since  $x$  and  $y$  have the same sum, we have  $\sum_{i=1}^n (x_i - y_i) = 0$ . Also,  $\sum_{i=1}^n |x_i - y_i| = \sum_{i \in R_x} (x_i - y_i) + \sum_{i \in R_y} (y_i - x_i) + \sum_{i=1}^n (x_i - y_i)$ , where the last term is zero. Simplifying and combining terms we have  $\sum_{i=1}^n |x_i - y_i| = 2 \sum_{i \in R_x} (x_i - y_i)$ . Thus  $\sum_{i=1}^n |x_i - y_i| / 2 = \sum_{i \in R_x} (x_i - y_i)$  which is the total amount by which the incomes of the persons in  $R_x$  are larger in  $x$  than in  $y$ . Consider any set of transfers  $t_{ij} \geq 0$ ,  $i, j \in N$ ,  $i \neq j$ . For notational convenience we define  $t_{ii} = 0$  for  $i \in N$ . For  $A, B \subseteq N$ , we let  $t(A, B) = \sum_{i \in A} \sum_{j \in B} t_{ij}$ . The value  $t(A, B) \geq 0$  is the total amount transferred from the set  $A$  to the set  $B$ . In particular  $t(N, N) \geq t(R_x, N)$  where the first quantity is the total amount transferred in the system and the second is the total transferred from all the persons in  $R_x$ . Every person  $i \in R_x$  must transfer a net amount equal to  $x_i - y_i$ , so that the total out of  $i$  is at least that much. Therefore, summing for  $i \in R_x$ , we have

$$t(N, N) \geq t(R_x, N) \geq \sum_{i \in R_x} (x_i - y_i) = \sum_{i=1}^n |x_i - y_i| / 2$$

which establishes the second half of the Lemma.

Assume that the total amount transferred by the system is  $t(N, N) = \sum_{i=1}^n |x_i - y_i| / 2$ .

Any  $x$ - $y$  system transfer must satisfy the condition that the net amount transferred from  $R_x$  to  $R_y$  is exactly that amount. Thus we can conclude that

$$(A.1) \quad t(R_x, R_y) \geq \sum_{i=1}^n |x_i - y_i| / 2$$

But we also have,

$$(A.2) \quad t(N, N) = t(R_x, R_x) + t(R_x, R_y) + t(R_y, R_x) + t(R_y, R_y)$$

Combining the hypothesis, (A.1) , (A.2) and the fact that all the terms are non-negative, we can conclude that  $t(R_x, R_y) = \sum_{i=1}^n |x_i - y_i|/2$  and  $t(R_x, R_x) = t(R_y, R_x) = t(R_y, R_y) = 0$ . Thus all transfers are from persons in  $R_x$  to persons in  $R_y$ , and Property D holds.

Assume now that Property D holds. It is obvious that the set of donors is  $R_x$  and the set of recipients is  $R_y$ . Since Property D holds, all the transfers in the system are from  $i \in R_x$  to  $j \in R_y$ , thus we have that  $t(N, N) = t(R_x, R_y)$  and  $t(R_x, R_x) = t(R_y, R_x) = t(R_y, R_y) = 0$ . In any  $x$ - $y$  system of transfers, the net amount transferred from  $R_x$  to  $R_y$  must equal the total amount by which the incomes of the persons in  $R_x$  are larger in  $x$  than in  $y$ , or mathematically,  $t(R_x, R_y) - t(R_y, R_x) = \sum_{i=1}^n |x_i - y_i|/2$ . Since  $t(R_y, R_x) = 0$ , we have

$$\sum_{i=1}^n |x_i - y_i|/2 = t(R_x, R_y) = t(N, N).$$

The proof of Theorem 2.1 employs the following Lemma.

**Lemma A.1.** If  $x \triangleleft y$ , then  $\sum_{i=k+1}^n x_{(i)} \geq \sum_{i=k+1}^n y_{(i)}$  for all  $k = 1, 2, \dots, n-1$  and in particular,

$$x_{(n)} \geq y_{(n)}$$

**Proof.** If  $x \triangleleft y$ , then  $\sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)}$  for all  $k=1,2,\dots,n-1$  and  $-\sum_{i=1}^n x_{(i)} \leq -\sum_{i=1}^n y_{(i)}$ .

Summing gives the first desired result. When  $k=n-1$ , we have  $x_{(n)} \geq y_{(n)}$  and the second part of the result is immediate as  $y_{(n)} \geq y_i$  for  $i=1,2,\dots,n$ .

### Proof of Theorem 2.1.

By deleting all persons with unchanged income a priori, we can assume that  $n = d(x, y)$ . Assume *w.l.g.* that  $y$  is increasingly ordered.

**Case I:**  $|R_x|=1$ . Assume that  $R_x = \{r\}$ . It is clear that transferring the amount  $y_i - x_i$  from  $r$  to  $i$  for all  $i \in R_y$  is a system of  $n-1$  transfers where the only donor is person  $r$  and therefore Property D holds. We will show that if these amounts are transferred first to the person in  $R_y$  who has the largest income in  $x$ , and then to the person in  $R_y$  who has the second largest income in  $x$ , and so on, then the sequence is a Muirhead system. Since  $x_i < y_i$  for all  $i \in R_y$ , and  $x_r > y_r$ , by Lemma A.1 we have that  $x_{(n)} = x_r$ , and  $x_r \geq y_i$  for all  $i \in N$ , that is the donor is the richest person. Choose the recipient  $k$  to be the richest person in  $R_y$ , that is,  $x_{(n-1)} = x_k$ . Transfer the amount  $\delta = y_k - x_k$  from  $r$  to  $k$ . It is clear that after this transfer, person  $k$  will have his final income of  $y_k$ , and since  $x_r \geq y_k$  this is a Muirhead transfer. The income of donor  $r$  will drop to  $x_r - (y_k - x_k)$ . We now show that the income of person  $r$  after the transfer is still the largest with the possible exception of person  $k$ . Re-arranging the equality  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$  we have

$$(A.3) \quad x_r - (y_k - x_k) = y_r + \sum_{i \in N \setminus \{k, r\}} (y_i - x_i).$$

But since  $x_{(n)} = x_r$  and  $x_{(n-1)} = x_k$ , we have  $\sum_{i \in N \setminus \{k, r\}} x_i = \sum_{i=1}^{n-2} x_{(i)}$ . For  $q \in N \setminus \{k\}$  we have that  $y_r + \sum_{i \in N \setminus \{k, r\}} y_i = y_q + \sum_{i \in N \setminus \{k, q\}} y_i \geq y_q + \sum_{i=1}^{n-2} y_{(i)}$  where the last inequality follows from the fact that the sum of a subset of size  $n-2$  of the vector  $y$  is bounded below by the sum of the  $n-2$  smallest elements. Combining the equality for  $x$  and inequality for  $y$  with (A.3) we have

$$(A.4) \quad x_r - (y_k - x_k) \geq y_q + \sum_{i=1}^{n-2} y_{(i)} - \sum_{i=1}^{n-2} x_{(i)} \geq y_q \quad \text{for all } q \in N \setminus \{k\}$$

where the last inequality in (A.4) follows from Definition 1.1. Since  $y_i > x_i$  for all  $i \in N \setminus \{k, r\}$ , we have

$$(A.5) \quad x_r - (y_k - x_k) > x_i \quad \text{for all } i \in N \setminus \{k, r\} \quad \text{and} \quad x_r - (y_k - x_k) \geq y_r.$$

We can now replace  $x$  by  $x'$  where  $x'_i = x_i$  for  $i \neq r$  and  $x'_r = x_r - (y_k - x_k)$ . Since person  $k$  has reached his final income  $y_k$ , we can delete component  $k$  from consideration and obtain two vectors  $x^*, y^* \in \mathfrak{R}^{n-1}$ ,  $x^* = (x'_1, \dots, x'_{k-1}, x'_{k+1}, \dots, x'_n)$  and  $y^* = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$ . It is clear that if  $x^* \neq y^*$  there is only one person with income larger in  $x^*$  than in  $y^*$ , and the opposite holds for the other persons. By (A.5) we have that

$$(A.6) \quad x_{(i)}^* = x_{(i)} \quad \text{for } i = 1, 2, \dots, n-2.$$

Also, it is clear that

$$(A.7) \quad \sum_{i=1}^p y_{(i)} \leq \sum_{i=1}^p y_{(i)}^* \quad \text{for } p = 1, 2, \dots, n-2.$$

Combining (A.6), (A.7) and the fact that  $x \triangleleft y$ , we have

$$(A.8) \quad \sum_{i=1}^p x_{(i)}^* = \sum_{i=1}^p x_{(i)} \leq \sum_{i=1}^p y_{(i)} \leq \sum_{i=1}^p y_{(i)}^* \quad \text{for } p = 1, 2, \dots, n-2.$$

To account for the deleted person  $k$  and the amount transferred from person  $r$

we have

$$(A.9) \quad \sum_{i=1}^{n-1} x_i^* = \left( \sum_{i=1}^n x_i \right) - (x_k + (y_k - x_k)) = \left( \sum_{i=1}^n x_i \right) - y_k = \left( \sum_{i=1}^n y_i \right) - y_k = \sum_{i=1}^{n-1} y_i^*.$$

Thus by (A.8) and (A.9),  $x^* \triangleleft y^*$ . Unless  $x^* = y^*$  the pair  $x^*$  and  $y^*$  has exactly one person with income larger in  $x^*$  than in  $y^*$ , so the procedure can be reapplied. Note that after  $n-1$  transfers are applied, the resulting  $x^*$  and  $y^*$  will both be vectors of size one and by the fact that  $x^*$  and  $y^*$  have the same sum throughout the procedure,  $x^* = y^*$ . Therefore, the number of transfers is  $n-1$ .

**Case II:**  $|R_y|=1$ . Assume that  $R_y = \{k\}$ . It is clear that transferring the amount  $x_i - y_i$  from  $i$  to  $k$  for all  $i \in R_x$  is a system of  $n-1$  transfers where the only recipient is person  $k$  and thus Property D holds. We will show that if these amounts are transferred first from the poorest person in  $R_x$ , followed by the next poorest, and so on, then the sequence is a sequence of Muirhead transfers. Since  $x_i > y_i$  for all  $i \in R_x$  and  $x_k < y_k$ , by Definition 1.1 we have that  $x_k = x_{(1)} \leq y_{(1)}$ , that is, the recipient is the poorest person. Choose the donor  $r$  to be the poorest person in  $R_x$ , that is,  $x_{(2)} = x_r$ . Transfer the amount  $\delta = x_r - y_r$  from  $r$  to  $k$ . It is clear that after this transfer, person  $r$  will have his final income of  $y_r$  and the income of recipient  $k$  will increase to  $x_k + (x_r - y_r)$ . Since  $x_k \leq y_{(1)} \leq y_r$  holds, by adding  $(x_r - y_r)$  we obtain  $x_k + (x_r - y_r) \leq x_r$  so this is a Muirhead transfer. We now show that after the transfer, the income of person  $k$  is still the smallest with the possible exception of person  $r$ . Rearranging the equality  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$  we have

$$(A.10) \quad x_k + (x_r - y_r) = y_k + \sum_{i \in N \setminus \{r, k\}} (y_i - x_i).$$

But since  $x_{(1)} = x_k$  and  $x_{(2)} = x_r$ , we have  $\sum_{i \in N \setminus \{r, k\}} x_i = \sum_{i=3}^n x_{(i)}$ . For  $q \in N \setminus \{r\}$  we have that

$y_k + \sum_{i \in N \setminus \{r, k\}} y_i = y_q + \sum_{i \in N \setminus \{r, q\}} y_i \leq y_q + \sum_{i=3}^n y_{(i)}$ , where the last inequality follows from the fact that the sum of a subset of size  $n-2$  of the vector  $y$  is bounded above by the sum of the  $n-2$  largest elements. Combining the equality for  $x$  and the inequality for  $y$  with (A.10) we have

$$(A.11) \quad x_k + (x_r - y_r) \leq y_q + \sum_{i=3}^n y_{(i)} - \sum_{i=3}^n x_{(i)} \leq y_q \quad \text{for all } q \in N \setminus \{r\}$$

where the last inequality in (A.11) follows from Lemma A.1. Since  $y_i < x_i$  for all  $i \in N \setminus \{k, r\}$ , we have

$$(A.12) \quad x_k + (x_r - y_r) < x_i \quad \text{for all } i \in N \setminus \{k, r\} \quad \text{and} \quad x_k + (x_r - y_r) \leq y_k.$$

We can now replace  $x$  by  $x'$  where  $x'_i = x_i$  for  $i \neq k$  and  $x'_k = x_k + (x_r - y_r)$ . Since person  $r$  has reached his final income  $y_r$ , we can delete component  $r$  from consideration and obtain two vectors  $x^*, y^* \in \mathfrak{R}^{n-1}$ , with  $x^* = (x'_1, \dots, x'_{r-1}, x'_{r+1}, \dots, x'_n)$  and  $y^* = (y_1, \dots, y_{r-1}, y_{r+1}, \dots, y_n)$ . It is clear that if  $x^* \neq y^*$ , there is only one person who has less income in  $x^*$  than in  $y^*$ , and the opposite holds for the other persons. By (A.12) we have that

$$(A.13) \quad x_{(1)}^* = x_{(1)} + (x_{(2)} - y_r) \quad \text{and} \quad x_{(i)}^* = x_{(i+1)} \quad \text{for } i = 2, 3, \dots, n-1.$$

Since  $y$  is increasingly ordered,  $(y_{(1)}^*, \dots, y_{(n-1)}^*) = (y_1, \dots, y_{r-1}, y_{r+1}, \dots, y_n)$ . Using (A.11) we can conclude that  $x_{(1)}^* \leq y_{(1)}^*$ . By (A.13) and Definition 1.1 we have that

$$\sum_{i=1}^p x_{(i)}^* = \left( \sum_{i=1}^{p+1} x_{(i)} \right) - y_r \leq \left( \sum_{i=1}^{p+1} y_i \right) - y_r \leq \sum_{i=1}^p y_{(i)}^* \quad \text{for } 2 \leq p \leq n-2.$$

Note that the last inequality is an equality when  $r \leq p+1$ , and otherwise, *i.e.* when  $r > p+1$ , the inequality holds since as  $y$  is increasing,  $y_{p+1} - y_r \leq 0$ . Therefore we have  $x^* \triangleleft y^*$ . We can now employ the same arguments as in Case I to show that

the procedure will require exactly  $n-1$  steps to reach  $y$  from  $x$ .

### **Proof of Theorem 2.2.**

Note that  $x^\diamond$  and  $y^\diamond$  have integer elements. We first restrict the search space by examining only integer-valued transfers. Since  $n=4$ , the total number of possible pairs  $(j,k)$  is 6. We will enumerate all the possible vectors  $x^1$  that can be reached from  $x^\diamond$  in one step where  $x^1 \triangleleft y^\diamond$  and then argue that the number of additional transfers is at least three. Note that if  $x^1 \triangleleft y^\diamond$  does hold, then  $y^\diamond$  cannot be reached from  $x^1$  and thus  $x^1$  need not be considered. In order to argue that the number of additional transfers is at least three, we will consider the vector  $d = \tilde{x} - y^\diamond$  where  $\tilde{x}$  is an intermediate vector, obtained from  $x^\diamond$  after a sequence of transfers (reaching  $y^\diamond$  is equivalent to sequentially reducing the number of non-zero components of  $d$  to zero). Since a Muirhead transfer involves two indices, at most two components of the vector  $d$  can become zero after one transfer. But a simple algebraic argument can be employed to show that a necessary condition for two components of  $d$  to become zero after a transfer of  $\delta > 0$  from  $j$  to  $k$  is that  $d_j + d_k = 0$ . Thus, if no pair  $(j,k)$  satisfies this condition, the number of components that can become zero is at most one. Furthermore, if the number of non-zero components of  $d$  is 3 at least two additional transfers are needed. Combining the two observations we can conclude that if the number of non-zeros component of the vector  $d$  is 4 and there is no pair  $(j,k)$  with the property  $d_j + d_k = 0$ , the number of additional transfers is at least three. We enumerate all the feasible values of  $x^1$  and summarize the result in Table A.1:

Table A.1

From ( $j$ )	To ( $k$ )	Units ( $\delta$ )	$x^1$	$d$	# additional transfers needed
1	2	1	19, 16, 1, 5	17, 12, -15, -14	$\geq 3$
1	2	5	15, 20, 1, 5	13, 16, -15, -14	$\geq 3$
1	3	19	1, 15, 20, 5	-1, 11, 4, -14	$\geq 3$
1	4	15	5, 15, 1, 20	3, 11, -15, 1	$\geq 3$
2	3	14	20, 1, 15, 5	18, -3, -1, -14	$\geq 3$
2	4	10	20, 5, 1, 15	18, 1, -15, -4	$\geq 3$
4	3	1	20, 15, 2, 4	18, 11, -14, -15	$\geq 3$
4	3	4	20, 15, 5, 1	18, 11, -11, -18	$\geq 2$

Table A.1 shows that in all but the last case the number of additional transfers that is needed is at least three (so the total is at least four). To complete the proof we need to show that despite the fact that  $x^1$  in the last case satisfies the necessary condition  $d_j + d_k = 0$ , the vector  $(2, 4, 16, 19)$  cannot be reached from  $x^1 = (20, 15, 5, 1)$  in two transfers. Following the arguments we stated above, in order to employ a total of two transfers when the number of non-zeros of  $d$  is 4, we need to use a pair  $(j, k)$  that satisfies the property  $d_j + d_k = 0$ .

*Case a:*  $j = 2, k = 3$ . In this case  $\delta$  can be at most 10 and since  $y^\diamond = (2, 4, 16, 19)$ , we must transfer 11 units from 2 to 3 in order for  $d_2 = d_3 = 0$ , thus at least three additional transfers are needed in order to reach  $y^\diamond$  from  $x^1$ .

Case b:  $j=1, k=4, \delta=18$ . The resulting vector is  $x^2 = (2,15,5,19)$ . However, as the property  $x^2 \triangleleft y^\diamond$  does not hold,  $y^\diamond$  cannot be reached from  $x^2$ , so  $x^2$  cannot be considered. Therefore, at least 3 additional transfers are needed in order to reach  $y^\diamond$  from  $x^1$ .

Note that the above example considered only integral values for the number of units transferred,  $\delta$ . It is possible to show that in general, if both  $y^\diamond$  and  $x^\diamond$  have only integral components and  $y^\diamond$  can be reached from  $x^\diamond$  in  $m$  transfers, one can construct a sequence of  $m$  transfers where all the intermediate vectors have only integral components. However, for the sake of brevity we will not prove this here, and it is not absolutely necessary. The example under consideration has the property that the set of choices for feasible  $x^1$  is fairly restricted and thus one can consider all real values for  $\delta$ . This change will only apply to the first and seventh cases, where instead of  $\delta = 1$ , one will have to consider  $\delta \in (0,1]$ . In the first case,  $x^1 = (20-\delta, 15+\delta, 1, 5)$  and  $d = (18-\delta, 11+\delta, -15, -14)$ . In the seventh case,  $d = (18, 11, -15+\delta, -14-\delta)$ . In both cases when  $\delta \in (0,1]$ , the number of non-zeros of  $d$  is 4 and there does not exist a pair  $(j,k)$  that satisfies the property  $d_j + d_k = 0$ . Thus the conclusion remains the same: the number of additional transfers is at least three. ■

Table A.2 contains all basic feasible solutions of  $TS(x,y)$  when  $x = (20,15,1,5)$  and  $y = (2,4,16,19)$ , that is, all  $x$ - $y$  systems of transfers that satisfy property NT. The table lists the arcs  $(i,j)$  and their flows. By simple enumeration one can verify that none of the solutions below is a Muirhead  $x$ - $y$  system of transfers. The table also gives the total amount transferred (See Lemma 2.1). The total amount by which

the persons in  $R_x$  have more income in  $x$  than in  $y$  is 29, so that only solutions 7 and 8 satisfy Property D.

Table A.2

Solution	Arc 1		Arc 2		Arc 3		Total Flow
	$(i, j)$	flow	$(i, j)$	flow	$(i, j)$	flow	
1	(1,2)	3	(1,3)	15	(2,4)	14	32
2	(1,2)	4	(1,4)	14	(2,3)	15	33
3	(1,2)	18	(2,3)	15	(2,4)	14	47
4	(1,2)	18	(2,3)	29	(3,4)	14	61
5	(1,2)	18	(2,4)	29	(4,3)	15	62
6	(1,3)	15	(1,4)	14	(2,1)	11	40
7	(1,3)	4	(1,4)	14	(2,3)	11	29 (*)
8	(1,3)	15	(1,4)	3	(2,4)	11	29 (*)
9	(1,3)	29	(2,1)	11	(3,4)	14	54
10	(1,3)	18	(2,3)	11	(3,4)	14	43
11	(1,3)	18	(2,4)	14	(3,2)	3	35
12	(1,3)	18	(2,4)	11	(3,4)	3	32
13	(1,4)	29	(2,1)	11	(4,3)	15	55
14	(1,4)	18	(2,3)	15	(4,2)	4	37
15	(1,4)	18	(2,3)	11	(4,3)	4	33
16	(1,4)	18	(2,4)	11	(4,3)	15	44

**REFERENCES**

Bossert, W. and M. Fleurbaey (2002): "Equitable Insurance Premium Schemes," *Social Choice and Welfare*, 19, 113-125.

Fields, G. and J. Fei (1978): "On Equality Comparisons," *Econometrica*, 46, 2, 303-316.

Foster, J. (1985): "Inequality measurement," in *Fair Allocation*, ed. by H. P. Young. Proceedings of Symposia in Applied Mathematics, 33: American Mathematical Society, Providence, 31-68.

Garey, M. and D. Johnson (1979): *Computers and Intractability: a Guide to the Theory of NP-Completeness*, San Francisco: W.H. Freeman.

Hardy, G., J. Littlewood and G. Polya (1952): *Inequalities*, 2<sup>nd</sup> Ed., Cambridge: Cambridge University Press.

Hillier, F. and G. Lieberman (2001): *Introduction to Operations Research*, 7<sup>th</sup> Ed., New York: McGraw Hill.

Marshall, A. and I. Olkin (1979): *Inequalities: Majorization Theory and its Applications*, New York: Academic Press.

Muirhead, R. (1903): "Some Methods Applicable to Identities and Inequalities of Symmetric Algebraic Functions of  $n$  letters," *Proceedings of the Edinburgh Mathematical Society*, 21, 144-157.

Sen, A. (1997): *On Economic Inequality*, Oxford: Clarendon Press.

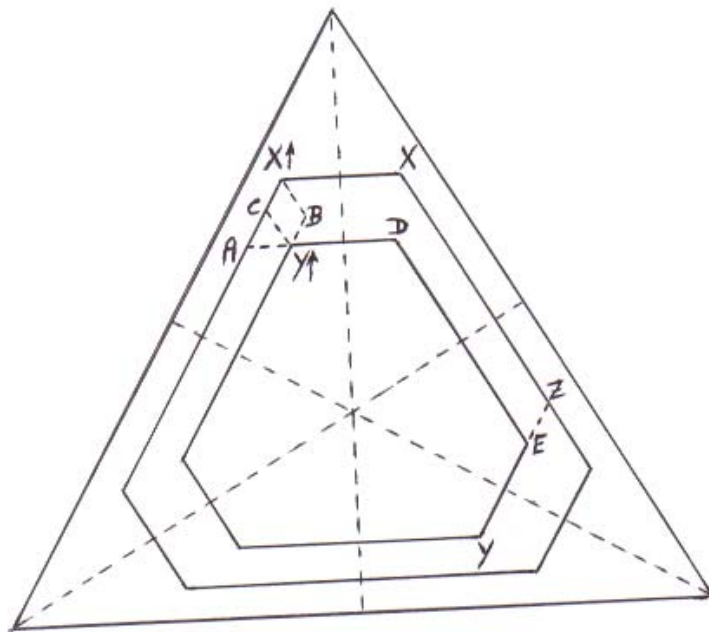


Figure 1

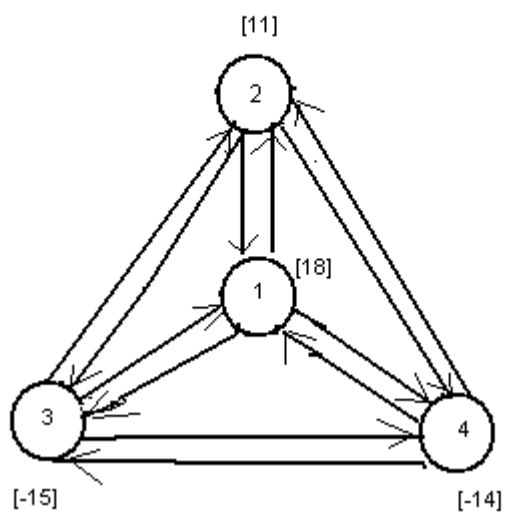


Figure 2  
TS(x,y), x=(20,15,1,5), y=(2,4,16,19),  
Supply in [ ].

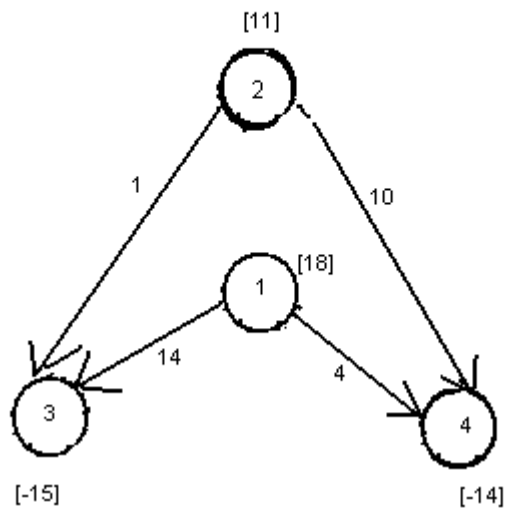


Figure 3  
Solution Z1, Flow shown on Arcs